Theory on Well-posedness of Boussinesq Equations with Fractional Laplacian

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Sub-critical Dissipation with Lipschitz Data

We consider the Cauchy problem of 2D fractional diffusion Boussinesq equations for an incompressible fluid flows in $\mathbb{R}^2$

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u + \nu (-\Delta)^{\alpha} u + \nabla P = \theta e_2, \\ \dfrac{\partial \theta}{\partial t} + (u \cdot \nabla) \theta + \kappa (-\Delta)^{\beta} \theta = 0, \\ \operatorname{div} u = 0, \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x). \end{cases} \tag{1}$$

where α, $\beta \in (0, 1)$, and $(-\Delta)^{\alpha}$ is the pseudodifferential operator defined via the Fourier transform

$$\widehat{(-\Delta)^{\alpha}} v(\xi) = |\xi|^{2\alpha} \widehat{v}(\xi).$$

In the following, for simplicity, we denote

$$\Lambda = (-\Delta)^{1/2}.$$

Theorem 3.4 (Global well-posedness; Xiaojing Xu 2010)

Let $\nu > 0$, $\kappa > 0$ be fixed, $\alpha \in \left[\frac{1}{2}, 1\right)$, $\beta \in \left(0, \frac{1}{2}\right]$, $\alpha + \beta = 1$, and $\mathrm{div}\, u_0 = 0$. Let $m > 2$ be an integer, and $(u_0, \theta_0) \in H^m\left(\mathbb{R}^2\right)$. Then, there exists a unique solution (u, θ) to the Cauchy problem (1) such that

$$\theta \in C([0,\infty); H^m(\mathbb{R}^2)) \cap L^2(0,\infty; H^{m+\beta}(\mathbb{R}^2)),$$

and

$$u \in C([0,\infty); H^m(\mathbb{R}^2)) \cap L^2(0,\infty; H^{m+\alpha}(\mathbb{R}^2)).$$

Remark 1.1. For simplicity of the exposition, we formulate and prove Theorem 3.4 in the subcritical case $\alpha + \beta = 1$, only. One can easily verify that, by arguments from this work, we can obtain an analogous result for $1 \leq \alpha + \beta \leq 2$.

Lemma 3.1 (Positive Inequality)

Let $0 \leq \alpha \leq 2$. For every $p > 1$, we have

$$\int_{\mathbb{R}^n} (\Lambda^\alpha w)\, |w|^{p-2} w \,\mathrm{d}x \geq C(p) \int_{\mathbb{R}^n} \left(\Lambda^{\frac{\alpha}{2}} |w|^{\frac{p}{2}}\right)^2 \,\mathrm{d}x, \qquad (2)$$

for all $w \in L^p(\mathbb{R}^n)$ such that $\Lambda^\alpha w \in L^p(\mathbb{R}^n)$, where $C(p) = \frac{4(p-1)}{p^2}$.

Remark: This inequality is well-known in the theory of sub-Markovian operators e.g.

- V.A. Liskevich, Yu.A. Semenov, *Some problems on Markov semigroups, Schrödinger operators, Markov semigroups, wavelet analysis, operator algebras*, Akademie Verlag, Berlin, 1996.

Observe that if $\alpha = 2$, integrating by parts we obtain (2) with the equality.

Theorem 3.5 (Blow-up Criterion; Xiaojing Xu 2010)

Let $\alpha, \beta \in [0,2]$, $\nu \geq 0$, $\kappa \geq 0$. Suppose $(u_0, \theta_0) \in H^m(\mathbb{R}^2)$ with $m > 2$ being an integer. Then, there exists a unique local classical solution $(u, \theta) \in C([0,T); H^m(\mathbb{R}^2))$ of problem (1) for some $T = T(\|u_0\|_{H^m(\mathbb{R}^2)}, \|\theta_0\|_{H^m(\mathbb{R}^2)})$. Moreover, the solution remains in $H^m\left(\mathbb{R}^2\right)$ up to a time $T_1 > T$, namely $(u, \theta) \in C\left([0, T_1]; H^m\left(\mathbb{R}^2\right)\right)$ if and only if

$$\int_0^T \|\nabla\theta(\tau)\|_{L^\infty} \mathrm{d}\tau < \infty. \tag{3}$$

• In the inviscid case $\nu = 0$ and $\kappa = 0$, Blow up Criterion was proved in Chae1997. The arguments from that works with minor changes also for problem (1) with the fractional diffusion, due to inequality (2). By this reason, we skip details of the proof of Theorem 3.5.
• In order to prove Theorem 3.4, it suffices to show that (3) holds true for the smooth solutions (u, θ) to the Cauchy problem (1).
• In the following section, we first show some a priori estimates for a smooth solution $(u, \theta) \in C([0, T); H^m(\mathbb{R}^2))$ with $m > 2$ to (1), then prove that (3) is valid.

For simplicity, let $\nu = \kappa = 1$.

- Estimate of $\|\theta\|_{L^\infty(0,\infty;\,L^p(\mathbb{R}^2))}$.

Let $p \geq 2$. Multiplying the second equation in (1) by $|\theta|^{p-2}\theta$ and integrating over $\mathbb{R}^2$, we deduce that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t)\|_{L^p}^p + \int_0^t (-\Delta)^\beta \theta |\theta|^{p-2}\theta \mathrm{d}x = 0,$$

where we have used the divergence free condition. This identity together with Lemma 3.1, allows us to get

$$\|\theta(t)\|_{L^p}^p + C(p)\int_0^t \|\Lambda^\beta |\theta|^{\frac{p}{2}}(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \leq \|\theta_0\|_{L^p}^p.$$

In particular, when $p = 2$, we have

$$\|\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\beta \theta(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \leq \|\theta_0\|_{L^2}^2. \tag{4}$$

- Estimate of $\|u\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))}$.

Multiplying the first equation of (1) by u, and integrating it over $\mathbb{R}^2$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2}^2 + \int_{\mathbb{R}^2} u(-\Delta)^\alpha u \,\mathrm{d}x = \int_{\mathbb{R}^2} \theta e_2 |u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^2} (u\cdot\nabla)|u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^2} \nabla P u \,\mathrm{d}x.$$

This identity together with inequality (2) and the divergence free condition, yield that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2}^2 + \int_{\mathbb{R}^2} |\Lambda^\alpha u|^2 \,\mathrm{d}x = \int_{\mathbb{R}^2} \theta e_2 u \,\mathrm{d}x \le \|\theta(t)\|_{L^2}\|u(t)\|_{L^2}.$$

By (4) and the Hölder inequality, we deduce that

$$\|u(t)\|_{L^2}^2 + 4\int_0^t \|\Lambda^\alpha u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \le 4\|\theta_0\|_{L^2}^2 T^2 + 2\|u_0\|_{L^2}^2.$$

- Estimate of $\|\omega(t)\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))}$.

Taking the operation curl on both sides of the first equation in (1) and denoting $\omega = \text{curl}\, u = \partial_{x_1} u_2 - \partial_{x_2} u_1$, we get

$$\omega_t + (-\Delta)^\alpha \omega + (u \cdot \nabla)\omega = -\theta_{x_1}. \tag{5}$$

Multiplying the above equality by ω, integrating over $\mathbb{R}^2$, we find

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t)\|_{L^2}^2 + \|\Lambda^\alpha \omega\|_{L^2}^2 &= \frac{1}{2}\int_{\mathbb{R}^2} (u \cdot \nabla)|\omega|^2 \,\mathrm{d}x - \int_{\mathbb{R}^2} \theta_{x_1}\omega \mathrm{d}x \\
&= -\int_{\mathbb{R}^2} \theta_{x_1}\omega \mathrm{d}x \\
&\le \frac{1}{2}\|\Lambda^\alpha \omega\|_{L^2}^2 + \frac{1}{2}\|\Lambda^\beta \theta\|_{L^2}^2
\end{aligned}$$

Here, in the last inequality, we have used the Parseval theorem and the relation $\alpha + \beta = 1$.

Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t)\|_{L^2}^2 + \|\Lambda^\alpha \omega\|_{L^2}^2 \leq \|\Lambda^\beta \theta\|_{L^2}^2.$$

By virtue of estimate (4), we deduce that

$$\begin{aligned}\|\omega(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\alpha \omega(\tau)\|_{L^2}^2 \, \mathrm{d}\tau &\leq \|\omega_0\|_{L^2}^2 + \int_0^t \|\Lambda^\beta \theta(\tau)\|_{L^2}^2 \, \mathrm{d}\tau \\ &\leq C\left(\|\omega_0\|_{L^2}, \|\theta_0\|_{L^2}, T\right).\end{aligned} \tag{6}$$

- Estimate of $\|\Lambda^\alpha D\omega\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))}$

We first compute the derivative $\nabla = (\partial_{x_1}, \partial_{x_2})$ of both sides of (5), and then take L^2 inner product with $\nabla\omega$. After integration by parts, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\omega(t)\|_{L^2}^2 + \|\Lambda^\alpha\nabla\omega\|_{L^2}^2 &= -\int_{\mathbb{R}^2}[\nabla(u\cdot\nabla)\omega]\nabla\omega\mathrm{d}x - \int_{\mathbb{R}^2}\nabla\theta_{x_1}\nabla\omega\mathrm{d}x \\
&= -\int_{\mathbb{R}^2}[(\nabla u\cdot\nabla)\omega]\nabla\omega\mathrm{d}x - \int_{\mathbb{R}^2}\nabla\theta_{x_1}\nabla\omega\mathrm{d}x \\
&\le \|\nabla u\|_{L^{\frac{2}{1-\alpha}}}\|\nabla\omega\|_{L^2}\|\nabla\omega\|_{L^{\frac{2}{\alpha}}} + \frac{1}{2}\|\Lambda^\alpha\nabla\omega\|_{L^2}^2 + \frac{1}{2}\|\Lambda^\beta\nabla\theta\|_{L^2}^2 \\
&\le \|\nabla u\|_{\frac{2}{1-\alpha}}\|\nabla\omega\|_{L^2}^{\frac{3\alpha-1}{\alpha}}\|\Lambda^\alpha\nabla\omega\|_{L^2}^{\frac{1-\alpha}{\alpha}} + \frac{1}{2}\|\Lambda^\alpha\nabla\omega\|_{L^2}^2 + \frac{1}{2}\|\Lambda^\beta\nabla\theta\|_{L^2}^2 \\
&\le C\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}^{\frac{2\alpha}{3\alpha-1}}\|\nabla\omega\|_{L^2}^2 + \frac{3}{4}\|\Lambda^\alpha\nabla\omega\|_{L^2}^2 + \frac{1}{2}\|\Lambda^\beta\nabla\theta\|_{L^2}^2
\end{aligned} \tag{7}$$

where we have used the assumptions $\alpha \ge \frac{1}{2}$ and $\operatorname{div} u = 0$.

Next, computing the derivative $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$ of the second equation from (1), we easily show that

$$\nabla^{\perp}\theta_t + \nabla^{\perp}[(u\cdot\nabla)\theta] + (-\Delta)^{\beta}\nabla^{\perp}\theta = 0.$$

We multiply the above equality by $\nabla^{\perp}\theta$, and integrate it over $\mathbb{R}^2$. Similar arguments as those in (7) lead to

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{\perp}\theta(t)\|_{L^2}^2 + \|\Lambda^{\beta}\nabla^{\perp}\theta\|_{L^2}^2 \\
\le& -\int_{\mathbb{R}^2}(u\cdot\nabla)\nabla^{\perp}\theta\nabla^{\perp}\theta \mathrm{d}x - \int_{\mathbb{R}^2}\nabla^{\perp}\theta\cdot\nabla u\nabla^{\perp}\theta \mathrm{d}x \\
=&\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}\|\nabla^{\perp}\theta\|_{L^2}\|\nabla^{\perp}\theta\|_{L^{\frac{2}{\alpha}}} \\
\le&\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}\|\nabla^{\perp}\theta\|_{L^2}\|\Lambda^{\beta}\nabla^{\perp}\theta\|_{L^2} \\
\le&\frac{1}{2}\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}^2\|\nabla^{\perp}\theta\|_{L^2}^2 + \frac{1}{2}\|\Lambda^{\beta}\nabla^{\perp}\theta\|_{L^2}^2.
\end{aligned} \tag{8}$$

Now, combining (7) with (8), one can show the function

$$X(t) = \|\nabla\omega(t)\|_{L^2} + \|\nabla^{\perp}\theta(t)\|_{L^2}.$$

satisfies the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) \le C(\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}^{\frac{2\alpha}{3\alpha-1}} + \|\nabla u\|_{L^{\frac{2}{1-\alpha}}}^{2})X(t).$$

Therefore, Gronwall's inequality and the embedding inequality and estimate (6) yield that

$$\begin{aligned}
X(t) &\le CX(0)\exp\{\int_0^t (\|\nabla u(\tau)\|_{L^{\frac{2}{1-\alpha}}}^{\frac{2\alpha}{3\alpha-1}} + \|\nabla u(\tau)\|_{L^{\frac{2}{1-\alpha}}}^{2})\mathrm{d}\tau\} \\
&\le CX(0)\exp\{\int_0^t (\|\Lambda^{\alpha}\omega(\tau)\|_{L^2}^{\frac{2\alpha}{3\alpha-1}} + \|\Lambda^{\alpha}\omega(\tau)\|_{L^2}^{2})\mathrm{d}\tau\} \\
&\le CX(0)\exp\left\{T^{\frac{4\alpha-2}{3\alpha-1}}\left(\int_0^t \|\Lambda^{\alpha}\omega(\tau)\|_{L^2}^{2}\ \mathrm{d}\tau\right)^{\frac{2\alpha}{3\alpha-1}} + \int_0^t \|\Lambda^{\alpha}\omega(\tau)\|_{L^2}^{2}\ \mathrm{d}\tau\right\} \\
&\le C(T, \|u_0\|_{H^2}, \|\theta_0\|_{H^1}).
\end{aligned}$$

Finally, by virtue of estimate (7), we deduce

$$\|\nabla\omega(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\alpha \nabla\omega(\tau)\|_{L^2}^2 \, \mathrm{d}\tau \leq C\left(T, \|u_0\|_{H^2}, \|\theta_0\|_{H^1}\right).$$

- Estimate of $\|\nabla\theta\|_{L^\infty(0,\infty;L^\infty(\mathbb{R}^2))}$

Multiplying (3.9) by $\left|\nabla^\perp\theta\right|^{p-2}\nabla^\perp\theta$, and integrating it over $\mathbb{R}^2$, we have

$$\begin{aligned}\frac{1}{p}\frac{d}{\mathrm{d}t}\|\nabla^\perp\theta(t)\|_{L^p}^p + C(p)\|\nabla^\perp\theta\|^p_{L^{\frac{2p}{1-\alpha}}} &\le \int_{\mathbb{R}^2}\nabla^\perp\theta\cdot\nabla u|\nabla^\perp\theta|^{p-2}\nabla^\perp\theta\mathrm{d}x\\ &\le \|\nabla u\|_{L^\infty}\|\nabla^\perp\theta\|_{L^p}^p.\end{aligned}$$

Using Gronwall's inequality and the obvious identity $\|\nabla Q\|_{L^p} = \|\nabla^\perp Q\|_{L^p}$, we easily show that

$$\|\nabla\theta(t)\|_{L^p} \le C\left\|\nabla\theta_0\right\|_{L^p}\exp\left\{\int_0^t\|\nabla u(\tau)\|_{L^\infty}\mathrm{d}\tau\right\}.$$

This inequality together with the Gagliardo-Nirenberg inequality, allows us to obtain

$$\begin{aligned}\int_0^t \|\nabla u(\tau)\|_{L^\infty}\mathrm{d}\tau &\le C\int_0^t \|\omega(\tau)\|_{L^2}^{\frac{\alpha}{1+\alpha}} \left\|\Lambda^{1+\alpha}\omega(\tau)\right\|_{L^2}^{\frac{1}{1+\alpha}}\,\mathrm{d}\tau \\ &\le CT^{\frac{2\alpha+1}{\alpha+1}}\left(\int_0^t \|\omega(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{\alpha}{1+2\alpha}} + C\int_0^t \left\|\Lambda^{1+\alpha}\omega(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau \\ &\le C\left(T, \|u_0\|_{H^2}, \|\theta_0\|_{H^1}\right).\end{aligned}$$

Using Sobolev embedding

$$\|f(x)\|_{L^p(\mathbb{R}^2)} \le C\|f(x)\|_{H^s(\mathbb{R}^2)}, \quad s > 1,$$

where C is independent of $p \in [2,\infty]$, then we have, for all $t \in [0, T]$,

$$\|\nabla\theta(t)\|_{L^p} \le C\,\|\theta_0\|_{H^m} \exp\left\{\int_0^t \|\nabla u(\tau)\|_{L^\infty}\mathrm{d}\tau\right\} \le C,$$

where C is independent of p.

Passing to the limit $p \to \infty$ in above inequality, we obtain

$$\|\nabla\theta(t)\|_{L^\infty} \le C\left(T, \|u_0\|_{H^2}, \|\theta_0\|_{H^m}\right), \quad \forall t \in [0, T].$$

This implies that condition (3) holds true, and according to Theorem 3.5, we obtain a unique solution of (1) such that $(u,\theta) \in C\left([0,\infty); H^m\left(\mathbb{R}^2\right)\right)$. By (7), (8) and the iteration process, we construct $u \in L^2\left(0,\infty; H^{m+\alpha}\left(\mathbb{R}^2\right)\right)$ and $\theta \in L^2\left(0,\infty; H^{m+\beta}\left(\mathbb{R}^2\right)\right)$, and we complete the proof of Theorem 3.4. $\square$

The Sub-critical Dissipation with Yudovich Type Data

Global well-posedness

Sub-critical Boussinesq Equations with Yudovich Initial Data

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu \Lambda^\alpha u = -\nabla p + \theta\, e_2, \\ \partial_t \theta + u \cdot \nabla \theta + \Lambda^\beta \theta = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{9}$$

where $\nu \geq 0$, $\alpha \in (0,1]$ and $\beta \in (1,2]$.

Theorem 1 (Wu-X., 2014)

Consider (9) with either $\nu > 0$ or $\nu = 0$, $\alpha \in (0,1]$ and $\beta \in (1,2]$. Let $q > \frac{2}{\beta-1}$. Assume u_0 satisfying $\nabla \cdot u_0 = 0$ and $u_0 \in L^2(\mathbb{R}^2)$, $\omega_0 \equiv \nabla \times u_0 \in L^q(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Assume $\theta_0 \in L^2(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$. Then, there exists a unique solution (u,θ) to (9) such that, for some $r \in (2,q)$,

$$u \in C_{loc}([0,\infty); L^2 \cap L^\infty), \quad \omega \in L^\infty_{loc}(0,\infty; L^q \cap L^\infty),$$
$$\theta \in C([0,\infty); L^2 \cap L^q) \cap L^2(0,\infty; H^{\frac{\beta}{2}}) \cap L^1_{loc}(0,\infty; B^{1+\frac{2}{r}}_{r,1}) \tag{10}$$

Furthermore, the bounds for (u,θ,ω) in the class (10) are independent of ν even in the case when $\nu > 0$.

Lemma 1

Assume that $\nabla \cdot u = 0$ and θ solves

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \Lambda^\beta \theta = 0, \\ \theta(x, 0) = \theta_0(x). \end{cases} \tag{11}$$

Let $r \in [2, \infty)$. Then θ obeys the estimate: for any integer $j \geq 0$,

$$2^{\beta j} \|\Delta_j \theta\|_{L_t^1 L^r} \leq C \, \|\theta_0\|_{L^r} \left(1 + \|\omega\|_{L_t^1 L^q} + (j+2) \|\omega\|_{L_t^1 L^\infty}\right). \tag{12}$$

Proof of Lemma. Applying Δ_j to (11) and taking inner product with $\Delta_j\theta|\Delta_j\theta|^{r-2}$, we obtain

$$\frac{1}{r}\frac{d}{dt}\|\Delta_j\theta\|_{L^r}^r+\int \Delta_j\theta\,|\Delta_j\theta|^{r-2}\Lambda^\beta\theta\,dx=-\int \Delta_j\theta|\Delta_j\theta|^{r-2}\,\Delta_j(u\cdot\nabla\theta)\,dx. \tag{13}$$

The dissipative term obeys the following lower bound

$$\int \Delta_j\theta\,|\Delta_j\theta|^{r-2}\Lambda^\beta\theta\,dx\geq C\,2^{\beta j}\|\Delta_j\theta\|_{L^r}^r.$$

Using the notion of para-products, we write

$$\Delta_j(u\cdot\nabla\theta) = J_{11} + J_{12} + J_{13} + J_{14} + J_{15},$$

where

$$\begin{aligned}
J_{11} &= \sum_{|j-k|\le 2} [\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k\theta,\\
J_{12} &= \sum_{|j-k|\le 2} (S_{k-1}u - S_j u)\cdot\nabla\Delta_j\Delta_k\theta,\\
J_{13} &= S_j u\cdot\nabla\Delta_j\theta,\\
J_{14} &= \sum_{|j-k|\le 2} \Delta_j(\Delta_k u\cdot\nabla S_{k-1}\theta),\\
J_{15} &= \sum_{k\ge j-1} \Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k\theta).
\end{aligned}$$

Since $\nabla \cdot u = 0$, we have

$$\int J_{13} |\Delta_j \theta|^{r-2} \Delta_j \theta \, dx = 0.$$

By Hölder's inequality,

$$\left| \int J_{11} |\Delta_j \theta|^{r-2} \Delta_j \theta \, dx \right| \leq \|J_{11}\|_{L^r} \|\Delta_j \theta\|_{L^r}^{r-1}.$$

Remark: Introduce the Littlewood-Paley decomposition, we write for each $j \in \mathbb{Z}$

$$A_j = \left\{ \xi \in \mathbb{R}^d : \; 2^{j-1} \leq |\xi| < 2^{j+1} \right\}.$$

The Littlewood-Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j \in \mathbb{Z}} \in \mathcal{S}$ such that

$$\mathrm{supp}\widehat{\Phi}_j \subset A_j, \qquad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x),$$

We write the commutator in terms of the integral,

$$J_{11} = \int \Phi_j(x-y)\,(S_{k-1}u(y) - S_{k-1}u(x)) \cdot \nabla \Delta_k \theta(y)\, dy,$$

where Φ_j is the kernel of the operator Δ_j. By Young's inequality for convolution,

$$\begin{aligned} \|J_{11}\|_{L^r} &\leq \||x|\Phi_j(x)\|_{L^1}\, \|\nabla S_{j-1}u\|_{L^\infty}\, \|\nabla \Delta_j \theta\|_{L^r} \\ &\leq \||x|\Phi_0(x)\|_{L^1}\, \|\nabla S_{j-1}u\|_{L^\infty}\, \|\Delta_j \theta\|_{L^r} \\ &= C\, \|\nabla S_{j-1}u\|_{L^\infty}\, \|\Delta_j \theta\|_{L^r}, \end{aligned}$$

where we have used the definition of Φ_j and Bernstein's inequality in the second inequality above.

By Bernstein's inequality,

$$\begin{aligned}
\|J_{12}\|_{L^r} &\leq C\|\Delta_j u\|_{L^\infty}\|\nabla\Delta_j\theta\|_{L^r}\\
&\leq C\,\|\nabla\Delta_j u\|_{L^\infty}\,\|\Delta_j\theta\|_{L^r}\\
&\leq C\,\|\omega\|_{L^\infty}\,\|\Delta_j\theta\|_{L^r};
\end{aligned}$$

$$\begin{aligned}
\|J_{14}\|_{L^r} &\leq C\|\Delta_j u\|_{L^\infty}\|\nabla S_{j-1}\theta\|_{L^r}\\
&\leq C\,\|\nabla\Delta_j u\|_{L^\infty}\sum_{m\leq j-2} 2^{(m-j)}\|\Delta_m\theta\|_{L^r}\\
&\leq C\,\|\omega\|_{L^\infty}\sum_{m\leq j-2} 2^{(m-j)}\|\Delta_m\theta\|_{L^r};
\end{aligned}$$

$$\begin{aligned}
\|J_{15}\|_{L^r} &\leq C\sum_{k\geq j-1} 2^{(j-k)}\,\|\nabla\Delta_k u\|_{L^\infty}\|\Delta_k\theta\|_{L^r}\\
&\leq C\,\|\omega\|_{L^\infty}\sum_{k\geq j-1} 2^{(j-k)}\,\|\Delta_k\theta\|_{L^r}.
\end{aligned}$$

Inserting the estimates above in (13), we have

$$\begin{aligned}\frac{d}{dt}\|\Delta_j\theta\|_{L^r} + C\,2^{\beta j}\|\Delta_j\theta\|_{L^r} \le C\,\|\nabla S_{j-1}u\|_{L^\infty}\,\|\Delta_j\theta\|_{L^r} \\ + C\,\|\omega\|_{L^\infty}\,[\|\Delta_j\theta\|_{L^r} + \sum_{m\le j-2} 2^{(m-j)}\|\Delta_m\theta\|_{L^r} \\ + \sum_{k\ge j-1} 2^{(j-k)}\,\|\Delta_k\theta\|_{L^r}].\end{aligned}$$

Integrating in time, taking L^1_t and multiplying by $2^{\beta j}$, we obtain (12). This completes the proof of Lemma 1.

Proof of Theorem 1

The local existence can be established through a standard procedure. We provide the global *a priori* bounds needed for the global existence. Obviously,

$$\|\theta(t)\|_{L^2} \le \|\theta_0\|_{L^2}, \quad \|\theta(t)\|_{L^q} \le \|\theta_0\|_{L^q},$$
$$\|u(t)\|_{L^2} \le \|u_0\|_{L^2} + t\,\|\theta_0\|_{L^2}.$$

Since ω satisfies

$$\partial_t \omega + u \cdot \nabla \omega + \nu \Lambda^\alpha \omega = \partial_{x_1} \theta,$$

it is clear that, for any $\nu \ge 0$,

$$\|\omega(t)\|_{L^q \cap L^\infty} \le \|\omega_0\|_{L^q \cap L^\infty} + \int_0^t \|\partial_{x_1} \theta(\tau)\|_{L^q \cap L^\infty}\, d\tau. \qquad (14)$$

To obtain a global bound for $\|\omega(t)\|_{L^q\cap L^\infty}$, we make use of the smoothing effect in the θ-equation. By Lemma 1, for any $2 < r < q$ and $0 < \epsilon < \beta$,

$$\sup_{j\geq -1} 2^{(\beta-\epsilon)j}\|\Delta_j\theta\|_{L^1_t L^r} \quad \leq \quad C\,\|\theta_0\|_{L^r}(1+\|\omega\|_{L^1_t(L^\infty\cap L^q)}).$$

Choosing $2 < r < q$ and $0 < \epsilon < \beta$ such that $1 + \frac{2}{r} < \beta - \epsilon$, we have, by Bernstein's inequality,

$$\|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}} \leq \sup_{j\geq -1} 2^{(\beta-\epsilon)j}\|\Delta_j\theta\|_{L^1_t L^r},$$

$$\begin{aligned} \|\nabla\theta(\tau)\|_{L^q\cap L^\infty} &\leq \sum_{j\geq -1} 2^j\|\Delta_j\theta\|_{L^q\cap L^\infty} \\ &\leq C \sum_{j\geq -1} 2^{(1+\frac{2}{r})j}\|\Delta_j\theta\|_{L^r} = C\,\|\theta\|_{B^{1+\frac{2}{r}}_{r,1}}. \end{aligned} \tag{15}$$

Combining these estimates, we obtain

$$\|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}} \leq C\,\|\theta_0\|_{L^r}\left(1 + t\|\omega_0\|_{L^\infty\cap L^q} + \int_0^t \|\theta\|_{L^1_\tau B^{1+\frac{2}{r}}_{r,1}}\,d\tau\right).$$

By Gronwall's inequality, for C depending on $\|\theta_0\|_{L^2\cap L^q}$ and $\|\omega_0\|_{L^\infty\cap L^q}$ only,

$$\|\theta\|_{L^1_t B^{1+\frac{2}{r}}_{r,1}} \leq C\,e^{C\,t}.$$

Consequently, by (14), $\|\omega\|_{L^\infty_t(L^\infty\cap L^q)} \leq C\,e^{C\,t}$.
In addition, by Gagliardo-Nirenberg inequality,

$$\|u\|_{L^\infty} \leq C\,\|u\|_{L^2}^{\frac{1}{2}}\,\|\omega\|_{L^\infty}^{\frac{1}{2}},$$

which yields the global bound for $\|u\|_{L^\infty}$. This completes the proof for the global bounds, which are independent of ν even in the case when $\nu > 0$.

Next we show that any two solutions satisfying (10) must coincide. Assume that (u_1, θ_1) and (u_2, θ_2) are two solutions of (9) and let p_1 and p_2 be the associated pressures, respectively. Consider the differences

$$u = u_2 - u_1, \quad \theta = \theta_2 - \theta_1, \quad p = p_2 - p_1,$$

which satisfy

$$\begin{aligned} &\partial_t u + u \cdot \nabla u_2 + u_1 \cdot \nabla u + \nu \Lambda^\alpha u = -\nabla p + \theta e_2, \\ &\partial_t \theta + u \cdot \nabla \theta_2 + u_1 \cdot \nabla \theta + \Lambda^\beta \theta = 0. \end{aligned}$$

Taking the inner product with (u, θ) and integrating by parts lead to

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + \|\theta\|_{L^2}^2\right) + \nu\|\Lambda^{\frac{\alpha}{2}} u\|_{L^2}^2 + \|\Lambda^{\frac{\beta}{2}}\theta\|_{L^2}^2 \\ \leq \|u\|_{L^2}\,\|\theta\|_{L^2} + J_1 + J_2, \tag{16}$$

where

$$J_1 = -\int u\cdot\nabla u_2 \cdot u\,dx, \qquad J_2 = -\int u\cdot\nabla\theta_2\cdot\theta\,dx.$$

For notational convenience, we let $\delta > 0$ and write

$$Y_\delta^2(t) = \delta^2 + \|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2.$$

For any $\rho \in [2, \infty)$, we have

$$|J_1| \leq \|u\|_{L^\infty}^{\frac{2}{\rho}} \|\nabla u_2\|_{L^\rho} \|u\|_{L^2}^{2-\frac{2}{\rho}}.$$

Furthermore,

$$\|u\|_{L^\infty} \leq \|u_1\|_{L^\infty} + \|u_2\|_{L^\infty} = L(t),$$
$$\|\nabla u_2\|_{L^\rho} \leq \rho \sup_{\rho\geq 2} \frac{\|\nabla u_2\|_{L^\rho}}{\rho} \leq \rho M(t)$$

where we have used the fact that $\sup_{\rho\geq 2} \frac{\|\nabla u_2\|_{L^\rho}}{\rho}$ is bounded due to $\omega_2 \in L^q \cap L^\infty$.

Therefore, by optimizing the bound over ρ, we have

$$|J_1| \leq M(t)\,\rho \left(\frac{L(t)}{Y_\delta}\right)^{\frac{2}{\rho}} Y_\delta^2 \leq 2e\,M(t)\,[\log L(t) - \log Y_\delta(t)]\;Y_\delta^2(t).$$

To bound J_2, we recall (15) to get

$$|J_2| \leq \|\nabla\theta_2\|_{L^\infty}\,\|u\|_{L^2}\,\|\theta\|_{L^2} \leq \|\theta_2\|_{B^{1+\frac{1}{r}}_{r,1}}\;Y_\delta^2(t).$$

Inserting the bounds above in (16), we obtain

$$\frac{d}{dt}Y_\delta(t) \leq 2e\,M(t)\,[\log L(t) - \log Y_\delta(t)]\;Y_\delta(t) + (1+\|\theta_2\|_{B^{1+\frac{1}{r}}_{r,1}})\;Y_\delta(t).$$

It then follows from applying the Osgood inequality that

$$Y_\delta(t) \le L(t)^{1-B(t)}\, Y_\delta(0)^{B(t)},$$

where

$$B(t) \equiv \exp\left(-\int_0^t (2e\, M(\tau) + \|\theta_2(\tau)\|_{B^{1+\frac{1}{r}}_{r,1}})\, d\tau\right).$$

Since $Y_\delta(0) = \delta$, we obtain by letting $\delta \to 0$ that

$$\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 \equiv 0$$

for any $t > 0$. This proves the uniqueness and thus Theorem 1.

The Critical Dissipation

The Cauchy problem of 2D fractional Boussinesq equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u + \nu\Lambda^{\alpha} u + \nabla P = \theta e_2, \\ \dfrac{\partial \theta}{\partial t} + (u \cdot \nabla)\theta + \kappa\Lambda^{\beta}\theta = 0, \\ \operatorname{div} u = 0, \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x). \end{cases} \tag{17}$$

where $u(t, x) = (u_1, u_2)$ is the velocity vector field, $P = P(t, x)$ is the scalar pressure, $\theta(t, x)$ is the scalar temperature, $\nu \geq 0$ is the viscosity, $\kappa \geq 0$ is the thermal diffusivity, $e_2 = (0, 1)$, α, $\beta \in (0, 2)$.

$$(\widehat{-\Delta)^{\alpha}} v(\xi) = |\xi|^{2\alpha} \widehat{v}(\xi).$$

In the following, for simplicity, we denote

$$\Lambda = (-\Delta)^{1/2}.$$

Two special critical cases

Hmidi, Keraani and Rousset were able to establish the global regularity of the critical Navier-Stokes-Boussinesq equations, namely (1) with $\nu > 0, \kappa = 0$ and $\alpha = 1$.

• T. Hmidi, S. Keraani and F. Rousset, Global well-posedness for a Boussinesq Navier-Stokes system with critical dissipation, J. Differential Equations 249 (2010).

Theorem 2 (Hmidi, Keraani and Rousset 2010)

Let $\theta^0 \in L^2 \cap B^0_{\infty,1}$ *and* v^0 *be a divergence-free vector field belonging to* $H^1 \cap \dot{W}^{1,p}$ *with* $p \in (2, +\infty)$. *Then the system has a unique global solution* (v, θ) *such that*

$$v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; H^1 \cap \dot{W}^{1,p}) \cap L^1_{\mathrm{loc}}(\mathbb{R}_+; B^1_{\infty,1})$$

and

$$\theta \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2 \cap B^0_{\infty,1}).$$

Motivation: Consider the new function

$$\omega - \mathcal{R}\theta.$$

The vorticity equation is given by

$$\omega_t + u \cdot \nabla\omega + \Lambda\omega = \theta_{x_1}, \qquad \Lambda = (-\Delta)^{\frac{1}{2}}.$$

The idea is to write $\Lambda\omega - \theta_{x_1} = \Lambda(\omega - \mathcal{R}\theta)$, $\mathcal{R} = \Lambda^{-1}\partial_{x_1}$ and consider the difference with $(\mathcal{R}\theta)_t + \overrightarrow{u} \cdot \nabla(\mathcal{R}\theta) = -[\mathcal{R}, u \cdot \nabla]\theta$,

$$(\omega - \mathcal{R}\theta)_t + u \cdot \nabla(\omega - \mathcal{R}\theta) + \nu\Lambda(\omega - \mathcal{R}\theta) = [\mathcal{R}, u \cdot \nabla]\theta.$$

The advantage of this new equation is that the commutator is much more regular and thus can be controlled. In fact, the L^q-norm of this commutator is more or less bound by $\|\nabla u\|_{L^q} \|\theta\|_{B^0_{\infty,1}}$ and thus this new formation makes the global L^q bound for ω possible.

They were also able to establish the global regularity of the critical Euler-Boussinesq equations, namely (1) with $\nu = 0, \kappa > 0$ and $\beta = 1$.

Theorem 3 (Hmidi, Keraani and Rousset 2011)

Let $p \in (2, \infty)$, $v^0 \in B^1_{\infty,1} \cap \dot{W}^{1,p}$ be a divergence-free vector field of $\mathbb{R}^2$ and $\theta^0 \in B^0_{\infty,1} \cap L^p$. Then there exists a unique global solution (v, θ) to the system with

$$v \in L^\infty_{loc}(\mathbb{R}_+; B^1_{\infty,1} \cap \dot{W}^{1,p}), \quad \theta \in L^\infty_{loc}(\mathbb{R}_+; B^0_{\infty,1} \cap L^p) \cap \widetilde{L}^1_{loc}(\mathbb{R}_+; B^1_{p,\infty}).$$

T. Hmidi, S. Keraani and F. Rousset, Global well-posedness for Euler-Boussinesq system with critical dissipation, Comm. Partial Differential Equations 36 (2011).

Proof of Theorem 3

Difficult

$$\|\omega\|_{L^2} \le C \quad ??$$

IDEA: Introduce a new function:

$$\omega + \mathcal{R}\theta.$$

Combine the equations of ω and $\mathcal{R}\theta$:

$$\omega_t + u \cdot \nabla\omega = \theta_{x_1}$$

$$(\mathcal{R}\theta)_t + u \cdot \nabla(\mathcal{R}\theta) = -\mathcal{R}\Lambda\theta - [\mathcal{R}, u \cdot \nabla]\theta$$

Then we obtain the following equation

$$(\omega + \mathcal{R}\theta)_t + u \cdot \nabla(\omega + \mathcal{R}\theta) = -[\mathcal{R}, u \cdot \nabla]\theta.$$

The main technical difficulty in this program when one takes the nonlinear terms into account is to evaluate the commutator $[\mathcal{R}, v\cdot\nabla]$. First we give some properties of the Riez operator $\mathcal{R}=\partial_1/|D|$. where the operator $|D|^\alpha$ is defined by

$$\mathcal{F}(|D|^\alpha u)(\xi)=|\xi|^\alpha(\mathcal{F}u)(\xi).$$

Proposition 4

Let $\mathcal{R}$ be the Riez operator $\mathcal{R}=\partial_1/|D|$. Then the following hold true.
i) For every $p\in(1,+\infty)$,

$$\|\mathcal{R}\|_{\mathcal{L}(L^p)}\lesssim 1.$$

Proposition 4

ii) Let $\chi \in \mathcal{D}(\mathbb{R}^d)$. Then, there exists $C > 0$ such that

$$\||\mathrm{D}|^s \chi(2^{-q}|\mathrm{D}|)\mathcal{R}\|_{\mathcal{L}(L^p)} \leq C2^{qs},$$

for every $(p, s, q) \in [1, \infty] \times (0, +\infty) \times \mathbb{N}$.
iii) Let $\mathcal{C}$ be a fixed ring. Then, there exists $\psi \in \mathcal{S}$ whose spectum does not meet the origin such that

$$\mathcal{R}f = 2^{qd}\psi(2^q \cdot) * f$$

for every f with Fourier transform supported in $2^q\mathcal{C}$.

Controlling the commutator between $\mathcal{R}$ and the convection operator $v \cdot \nabla$ is a crucial ingredient in the proof of Theorem 3.

Proposition 5 (Commutator estimate)

Let v be is a smooth divergence-free vector field.
i) For every $(p, r) \in [2, \infty) \times [1, \infty]$ there exists a constant $C = C(p, r)$ such that

$$\|[\mathcal{R}, v \cdot \nabla]\theta\|_{B^0_{p,r}} \leq C\|\nabla v\|_{L^p}\big(\|\theta\|_{B^0_{\infty,r}} + \|\theta\|_{L^p}\big),$$

for every smooth scalar function θ.
ii) For every $(r, \rho) \in [1, \infty] \times (1, \infty)$ and $\epsilon > 0$ there exists a constant $C = C(r, \rho, \varepsilon)$ such that

$$\|[\mathcal{R}, v \cdot \nabla]\theta\|_{B^0_{\infty,r}} \leq C(\|\omega\|_{L^\infty} + \|\omega\|_{L^\rho})\big(\|\theta\|_{B^\epsilon_{\infty,r}} + \|\theta\|_{L^\rho}\big),$$

for every smooth scalar function θ.

Now, turn to prove Theorem 3.(For simlicity, we just give *a priori* estimate)
First, we will give some estimates for the linear transport-diffusion model

$$\begin{cases} \partial_t\theta + v\cdot\nabla\theta + |\mathrm{D}|\theta = f \\ \theta_{|t=0} = \theta^0. \end{cases} \tag{TD}$$

We can easily get the L^p estimate, for $p \in [1, \infty]$

$$\|\theta(t)\|_{L^p} \le \|\theta^0\|_{L^p} + \int_0^t \|f(\tau)\|_{L^p} d\tau. \tag{18}$$

For $p \in [1, \infty)$ there exists a constant C such that

$$\sup_{q\in\mathbb{N}} 2^q \|\Delta_q\theta\|_{L^1_t L^p} \le C\|\theta^0\|_{L^p} + C\|\theta^0\|_{L^\infty}\|\omega\|_{L^1_t L^p}, \tag{19}$$

for every smooth solution θ of (TD) with $f = 0$.

Applying Riesz transform $\mathcal{R}$ to the temperature equation we get

$$\partial_t \mathcal{R}\theta + v \cdot \nabla \mathcal{R}\theta + |\mathrm{D}|\mathcal{R}\theta = -[\mathcal{R}, v \cdot \nabla]\theta. \tag{20}$$

Since $|\mathrm{D}|\mathcal{R} = \partial_1$ then the function $\Gamma := \omega + \mathcal{R}\theta$ satisfies

$$\partial_t \Gamma + v \cdot \nabla \Gamma = -[\mathcal{R}, v \cdot \nabla]\theta. \tag{21}$$

According to the first part of Proposition 6

$$\|[\mathcal{R}, v \cdot \nabla]\theta\|_{B^0_{p,r}} \leq C\|\nabla v\|_{L^p}(\|\theta\|_{B^0_{\infty,r}} + \|\theta\|_{L^p}),$$

applied with $r = 2$ we have

$$\big\|[\mathcal{R}, v \cdot \nabla]\theta\big\|_{B^0_{p,2}} \lesssim \|\nabla v\|_{L^p}\big(\|\theta\|_{B^0_{\infty,2}} + \|\theta\|_{L^p}\big).$$

Using the classical embedding $B^0_{p,2} \hookrightarrow L^p$ which is true only for $p \in [2, \infty)$

$$\big\|[\mathcal{R}, v \cdot \nabla]\theta\big\|_{L^p} \leq \|\nabla v\|_{L^p}\big(\|\theta\|_{B^0_{\infty,2}} + \|\theta\|_{L^p}\big).$$

Since $\operatorname{div} v = 0$ then we get from the transport equation (21) $(\partial_t \Gamma + v \cdot \nabla \Gamma = -[\mathcal{R}, v \cdot \nabla]\theta)$

$$\|\Gamma(t)\|_{L^p} \leq \|\Gamma^0\|_{L^p} + \int_0^t \|[\mathcal{R}, v \cdot \nabla]\theta(\tau)\|_{L^p} d\tau.$$

Putting together the last two estimates we get

$$\begin{aligned} \|\Gamma(t)\|_{L^p} &\lesssim \|\Gamma^0\|_{L^p} + \int_0^t \|\nabla v(\tau)\|_{L^p}\big(\|\theta(\tau)\|_{B^0_{\infty,2}} + \|\theta\|_{L^p}\big) d\tau \\ &\lesssim \|\omega^0\|_{L^p} + \|\theta^0\|_{L^p} + \int_0^t \|\omega(\tau)\|_{L^p}\big(\|\theta(\tau)\|_{B^0_{\infty,2}} + \|\theta^0\|_{L^p}\big) d\tau. \end{aligned}$$

On the other hand, from the continuity of the Riesz transform Proposition 4 $(\|\mathcal{R}\|_{\mathcal{L}(L^p)} \lesssim 1)$ and (18) $(\|\theta(t)\|_{L^p} \leq \|\theta^0\|_{L^p} + \int_0^t \|f(\tau)\|_{L^p} d\tau)$

$$\begin{aligned} \|\omega(t)\|_{L^p} &\leq \|\Gamma(t)\|_{L^p} + \|\mathcal{R}\theta\|_{L^p} \\ &\lesssim \|\Gamma(t)\|_{L^p} + \|\theta^0\|_{L^p}. \end{aligned}$$

This leads to

$$\|\omega(t)\|_{L^p} \lesssim \|\omega^0\|_{L^p} + \|\theta^0\|_{L^p} + \int_0^t \|\omega(\tau)\|_{L^p} \big(\|\theta(\tau)\|_{B^0_{\infty,2}} + \|\theta^0\|_{L^p}\big) d\tau.$$

According to Gronwall lemma we get

$$\|\omega(t)\|_{L^p} \leq C_0 e^{C_0 t} e^{C\|\theta\|_{L^1_t B^0_{\infty,2}}}. \tag{22}$$

Let $N \in \mathbb{N}$, by Bernstein inequalities and (18)

$$
\begin{aligned}
\|\theta\|_{L^1_t B^0_{\infty,2}} &\leq \|S_N\theta\|_{L^1_t B^0_{\infty,2}} + \|(\mathrm{Id} - S_N)\theta\|_{L^1_t B^0_{\infty,1}} \\
&\lesssim t\|\theta^0\|_{L^\infty}\sqrt{N} + \sum_{q\geq N} \|\Delta_q\theta\|_{L^1_t L^\infty} \\
&\lesssim \sqrt{N}\|\theta^0\|_{L^\infty} t + \sum_{q\geq N} 2^{q\frac{2}{p}} \|\Delta_q\theta\|_{L^1_t L^p}.
\end{aligned}
$$

Using (19) $(\sup_{q\in\mathbb{N}} 2^q\|\Delta_q\theta\|_{L^1_t L^p} \leq C\|\theta^0\|_{L^p} + C\|\theta^0\|_{L^\infty}\|\omega\|_{L^1_t L^p})$ and $p > 2$ we obtain

$$
\begin{aligned}
\sum_{q\geq N-1} 2^{q\frac{2}{p}} \|\Delta_q\theta\|_{L^1_t L^p} &\lesssim \sum_{q\geq N-1} 2^{q(\frac{2}{p}-1)}\Big(\|\theta^0\|_{L^p} + \|\theta^0\|_{L^\infty}\int_0^t \|\omega(\tau)\|_{L^p} d\tau\Big) \\
&\lesssim \|\theta^0\|_{L^p} + 2^{N(-1+\frac{2}{p})}\|\theta^0\|_{L^\infty}\int_0^t \|\omega(\tau)\|_{L^p} d\tau.
\end{aligned}
$$

Thus, we get

$$\|\theta\|_{L^1_t B^0_{\infty,2}} \lesssim \sqrt{N}\|\theta^0\|_{L^\infty} t + \|\theta^0\|_{L^p} + 2^{N(-1+\frac{2}{p})}\|\theta^0\|_{L^\infty} \int_0^t \|\omega(\tau)\|_{L^p} d\tau.$$

We choose N as follows

$$N = \left[\frac{\log\left(e + \int_0^t \|\omega(\tau)\|_{L^p} d\tau\right)}{(1-2/p)\log 2} \right] + 1.$$

Then it follows

$$\|\theta\|_{L^1_t B^0_{\infty,2}} \lesssim \|\theta^0\|_{L^\infty \cap L^p} + \|\theta^0\|_{L^\infty} t \log^{\frac{1}{2}}\left(e + \int_0^t \|\omega(\tau)\|_{L^p} d\tau\right).$$

Combining this estimate with (22) $(\|\omega(t)\|_{L^p} \le C_0 e^{C_0 t} e^{C\|\theta\|_{L^1_t B^0_{\infty,2}}})$ we get

$$\begin{aligned}\|\theta\|_{L^1_t B^0_{\infty,2}} &\lesssim \|\theta^0\|_{L^\infty \cap L^p} + \|\theta^0\|_{L^\infty} t \log^{\frac{1}{2}}\left(e + C_0 e^{C_0 t} e^{C\|\theta\|_{L^1_t B^0_{\infty,2}}}\right) \\ &\le C_0(1+t^2) + C\|\theta^0\|_{L^\infty} t \|\theta\|^{\frac{1}{2}}_{L^1_t B^0_{\infty,2}}.\end{aligned}$$

Thus we get for every $t \in \mathbb{R}+$

$$\|\theta\|_{L^1_t B^0_{\infty,2}} \le C_0(1+t^2).$$

It follows from (22) $(\|\omega(t)\|_{L^p} \le C_0 e^{C_0 t} e^{C\|\theta\|_{L^1_t B^0_{\infty,2}}})$

$$\|\omega(t)\|_{L^p} \le \Phi_1(t). \tag{23}$$

Applying (19) $(\sup_{q\in\mathbb{N}} 2^q \|\Delta_q\theta\|_{L^1_t L^p} \le C\|\theta^0\|_{L^p} + C\|\theta^0\|_{L^\infty}\|\omega\|_{L^1_t L^p})$ and (23) we get

$$2^q \|\Delta_q \theta\|_{L^1_t L^p} \le \Phi_1(t), \qquad \forall q \in \mathbb{N} \tag{24}$$

and thus

$$\|\theta\|_{\widetilde{L}^1_t B^1_{p,\infty}} \le \Phi_1(t).$$

Combing with (23), we have

$$\|\omega(t)\|_{L^p} + \|\theta\|_{\widetilde{L}^1_t B^1_{p,\infty}} \le \Phi_1(t), \;\; p \in (2,\infty). \tag{25}$$

It is not hard to see that from (24) $(2^q\|\Delta_q\theta\|_{L^1_t L^p} \leq \Phi_1(t))$ one can obtain that for every $s < 1$

$$\|\theta\|_{L^1_t B^s_{p,1}} \leq \|\theta\|_{\widetilde{L}^1_t B^1_{p,\infty}} \leq \Phi_1(t). \tag{26}$$

Combined with Bernstein inequalities and the fact that $p > 2$ this yields

$$\|\theta\|_{L^1_t B^{\epsilon}_{\infty,1}} \leq \Phi_1(t), \tag{27}$$

for every $\epsilon < 1 - \frac{2}{p}$.

By using the maximum principle for the transport equation (21) $(\partial_t \Gamma + v \cdot \nabla \Gamma = -[\mathcal{R}, v \cdot \nabla]\theta)$, we get

$$\|\Gamma(t)\|_{L^\infty} \le \|\Gamma^0\|_{L^\infty} + \int_0^t \|[\mathcal{R}, v \cdot \nabla]\theta(\tau)\|_{L^\infty} d\tau.$$

Since the function $\mathcal{R}\theta$ satisfies the equation

$$\big(\partial_t + v \cdot \nabla + |D|\big)\mathcal{R}\theta = -[\mathcal{R}, v \cdot \nabla]\theta, \tag{28}$$

we get by using (18) $(\|\theta(t)\|_{L^p} \le \|\theta^0\|_{L^p} + \int_0^t \|f(\tau)\|_{L^p} d\tau)$ for $p = \infty$ that

$$\|\mathcal{R}\theta(t)\|_{L^\infty} \le \|\mathcal{R}\theta^0\|_{L^\infty} + \int_0^t \|[\mathcal{R}, v \cdot \nabla]\theta(\tau)\|_{L^\infty} d\tau.$$

Combining the last two estimates yields

$$\begin{aligned} & \|\Gamma(t)\|_{L^\infty} + \|\mathcal{R}\theta(t)\|_{L^\infty} \\ \le\ & \|\Gamma^0\|_{L^\infty} + \|\mathcal{R}\theta^0\|_{L^\infty} + 2\int_0^t \|[\mathcal{R}, v\cdot\nabla]\theta(\tau)\|_{L^\infty} d\tau \\ \le\ & C_0 + \int_0^t \|[\mathcal{R}, v\cdot\nabla]\theta(\tau)\|_{B^0_{\infty,1}} d\tau. \end{aligned}$$

It follows from the second estimate of Proposition 6 $(\|[\mathcal{R}, v\cdot\nabla]\theta\|_{B^0_{\infty,r}} \le C(\|\omega\|_{L^\infty} + \|\omega\|_{L^\rho})(\|\theta\|_{B^\epsilon_{\infty,r}} + \|\theta\|_{L^\rho}))$ and (25) $(\|\omega(t)\|_{L^p} + \|\theta\|_{\widetilde{L}^1_t B^1_{p,\infty}} \le \Phi_1(t))$

$$\begin{aligned} & \|\omega(t)\|_{L^\infty} + \|\mathcal{R}\theta(t)\|_{L^\infty} \\ \lesssim\ & C_0 + \int_0^t \|\omega(\tau)\|_{L^\infty\cap L^p}\big(\|\theta(\tau)\|_{B^\epsilon_{\infty,1}} + \|\theta(\tau)\|_{L^p}\big) d\tau \\ \lesssim\ & C_0 + \|\omega\|_{L^\infty_t L^p}\big(\|\theta\|_{L^1_t B^\epsilon_{\infty,1}} + t\|\theta^0\|_{L^p}\big) \\ & + \int_0^t \|\omega(\tau)\|_{L^\infty}\big(\|\theta(\tau)\|_{B^\epsilon_{\infty,1}} + \|\theta^0\|_{L^p}\big) d\tau. \end{aligned}$$

Let $0 < \epsilon < 1 - \frac{2}{p}$ then using (27) $(\|\theta\|_{L^1_t B^{\epsilon}_{\infty,1}} \leq \Phi_1(t))$ we get

$$\|\omega(t)\|_{L^\infty} + \|\mathcal{R}\theta(t)\|_{L^\infty} \lesssim \Phi_1(t) + \int_0^t \|\omega(\tau)\|_{L^\infty} \left(\|\theta(\tau)\|_{B^{\epsilon}_{\infty,1}} + \|\theta^0\|_{L^p}\right) d\tau.$$

Therefore we obtain by the Gronwall lemma and a new use of (27) that

$$\|\omega(t)\|_{L^\infty} + \|\mathcal{R}\theta(t)\|_{L^\infty} \leq \Phi_2(t). \tag{29}$$

Let $N \in \mathbb{N}$ to be chosen later. Using the fact that $\|\dot{\Delta}_q v\|_{L^\infty} \approx 2^{-q}\|\dot{\Delta}_q \omega\|_{L^\infty}$, we then have

$$\begin{aligned} \|v(t)\|_{L^\infty} &\leq \|\chi(2^{-N}|D|)v(t)\|_{L^\infty} + \sum_{q \geq -N} 2^{-q}\|\dot{\Delta}_q \omega(t)\|_{L^\infty} \\ &\leq \|\chi(2^{-N}|D|)v(t)\|_{L^\infty} + 2^N\|\omega(t)\|_{L^\infty}. \end{aligned}$$

Applying the frequency localizing operator to the velocity equation we get

$$\begin{aligned}\chi(2^{-N}|D|)v =&\chi(2^{-N}|D|)v_0 + \int_0^t \mathcal{P}\chi(2^{-N}|D|)\theta(\tau)d\tau \\ &+ \int_0^t \mathcal{P}\chi(2^{-N}|D|)\mathrm{div}(v\otimes v)(\tau)d\tau.\end{aligned}$$

where $\mathcal{P}$ stands for Leray projector. From Bernstein inequalities, Calderón-Zygmund estimate and the uniform boundness of $\chi(2^{-N}|D|)$ we get

$$\begin{aligned}\int_0^t \|\chi(2^{-N}|D|)\mathcal{P}\theta(\tau)\|_{L^\infty}d\tau &\lesssim 2^{-N\frac{2}{p}}\int_0^t \|\theta(\tau)\|_{L^p}d\tau \\ &\lesssim t\|\theta^0\|_{L^p}.\end{aligned}$$

Using Proposition 4-(ii) $(\||\mathrm{D}|^s\chi(2^{-q}|\mathrm{D}|)\mathcal{R}\|_{\mathcal{L}(L^p)} \leq C2^{qs})$ we find

$$\int_0^t \|\mathcal{P}\chi(2^{-N}|D|)\mathrm{div}(v\otimes v)(\tau)\|_{L^\infty}d\tau \lesssim 2^{-N}\int_0^t \|v(\tau)\|_{L^\infty}^2 d\tau.$$

The outcome is

$$\begin{aligned}\|v(t)\|_{L^\infty} &\lesssim \|v^0\|_{L^\infty} + t\|\theta_0\|_{L^p} + 2^{-N}\int_0^t \|v(\tau)\|_{L^\infty}^2 d\tau + 2^N\|\omega(t)\|_{L^\infty}\\ &\lesssim 2^{-N}\int_0^t \|v(\tau)\|_{L^\infty}^2 d\tau + 2^N\Phi_2(t)\end{aligned}$$

Choosing judiciously N we find

$$\|v(t)\|_{L^\infty} \leq \Phi_2(t)\Big(1+\Big(\int_0^t \|v(\tau)\|_{L^\infty}^2 d\tau\Big)^{\frac{1}{2}}\Big).$$

From Gronwall lemma we get

$$\|v(t)\|_{L^\infty} \leq \Phi_3(t). \tag{30}$$

First, we need a logarithmic estimates of the transport equation

$$(\partial_t + v\cdot\nabla + \kappa|D|)\theta = f,$$

then we have

$$\|\theta\|_{\widetilde{L}_t^\infty B_{p,1}^0} \leq C\Big(\|\theta^0\|_{B_{p,1}^0} + \|f\|_{L_t^1 B_{p,1}^0}\Big)\Big(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\Big), \tag{31}$$

for $p \in [1,\infty]$(Details see Theorem 4.5 in Hmidi,Keraani and Rousset 2011).

By using the logarithmic estimates of the equations (21) $(\partial_t\Gamma + v\cdot\nabla\Gamma = -[\mathcal{R}, v\cdot\nabla]\theta)$ and (28) $((\partial_t + v\cdot\nabla + |D|)\mathcal{R}\theta = -[\mathcal{R}, v\cdot\nabla]\theta)$, we obtain

$$\|\Gamma(t)\|_{B^0_{\infty,1}} + \|\mathcal{R}\theta(t)\|_{B^0_{\infty,1}} \lesssim \Big(C_0 + \big\|[\mathcal{R}, v\cdot\nabla]\theta\big\|_{L^1_t B^0_{\infty,1}}\Big)\Big(1 + \|\nabla v\|_{L^1_t L^\infty}\Big). \tag{32}$$

Thanks to Propositions 6 $(\|[\mathcal{R}, v\cdot\nabla]\theta\|_{B^0_{\infty,r}} \leq C(\|\omega\|_{L^\infty} + \|\omega\|_{L^\rho})(\|\theta\|_{B^\epsilon_{\infty,r}} + \|\theta\|_{L^\rho}))$, (25)$(\|\omega(t)\|_{L^p} + \|\theta\|_{\widetilde{L}^1_t B^1_{p,\infty}} \leq \Phi_1(t))$ (29)$(\|\omega(t)\|_{L^\infty} + \|\mathcal{R}\theta(t)\|_{L^\infty} \leq \Phi_2(t))$ and (27) $(\|\theta\|_{L^1_t B^\epsilon_{\infty,1}} \leq \Phi_1(t))$ we get

$$\begin{aligned}\big\|[\mathcal{R}, v\cdot\nabla]\theta\big\|_{L^1_t B^0_{\infty,1}} \lesssim & \int_0^t (\|\omega(\tau)\|_{L^\infty} + \|\omega(\tau)\|_{L^p}) \\ & \big(\|\theta(\tau)\|_{B^\epsilon_{\infty,1}} + \|\theta(\tau)\|_{L^p}\big)\,d\tau \\ \lesssim & \Phi_2(t).\end{aligned}$$

By easy computations we get

$$\begin{aligned}
\|\nabla v\|_{L^\infty} &\leq \|\nabla\Delta_{-1}v\|_{L^\infty} + \sum_{q\in\mathbb{N}}\|\Delta_q\nabla v\|_{L^\infty} \\
&\lesssim \|\omega\|_{L^p} + \sum_{q\in\mathbb{N}}\|\Delta_q\omega\|_{L^\infty} \\
&\lesssim \Phi_1(t) + \|\omega(t)\|_{B^0_{\infty,1}}. \qquad (33)
\end{aligned}$$

Putting together (32) and (33) leads to

$$\begin{aligned}
\|\omega(t)\|_{B^0_{\infty,1}} &\leq \|\Gamma(t)\|_{B^0_{\infty,1}} + \|\mathcal{R}\theta(t)\|_{B^0_{\infty,1}} \\
&\leq \Phi_2(t)\Big(1 + \int_0^t \|\omega(\tau)\|_{B^0_{\infty,1}}\,d\tau\Big).
\end{aligned}$$

Thus we obtain from Gronwall inequality

$$\|\omega(t)\|_{B^0_{\infty,1}} \leq \Phi_3(t). \qquad (34)$$

Coming back to (33)$(\|\nabla v\|_{L^\infty} \lesssim \Phi_1(t) + \|\omega(t)\|_{B^0_{\infty,1}})$ we get

$$\|\nabla v(t)\|_{L^\infty} \leq \Phi_3(t).$$

Let us move to the estimate of v in the space $B^1_{\infty,1}$. By definition we have

$$\|v(t)\|_{B^1_{\infty,1}} \lesssim \|v(t)\|_{L^\infty} + \|\omega(t)\|_{B^0_{\infty,1}}.$$

Combined with (30) $(\|v(t)\|_{L^\infty} \leq \Phi_3(t))$ and (34) $(\|\omega(t)\|_{B^0_{\infty,1}} \leq \Phi_3(t))$ this yields

$$\|v(t)\|_{B^1_{\infty,1}} \leq \Phi_3(t).$$

Commutator Estimate

Proposition 6 (Commutator estimate)

Let v be is a smooth divergence-free vector field.
i) For every $(p, r) \in [2, \infty) \times [1, \infty]$ there exists a constant $C = C(p, r)$ such that

$$\|[\mathcal{R}, v \cdot \nabla]\theta\|_{B^0_{p,r}} \leq C\|\nabla v\|_{L^p}\big(\|\theta\|_{B^0_{\infty,r}} + \|\theta\|_{L^p}\big),$$

for every smooth scalar function θ.
ii) For every $(r, \rho) \in [1, \infty] \times (1, \infty)$ and $\epsilon > 0$ there exists a constant $C = C(r, \rho, \varepsilon)$ such that

$$\|[\mathcal{R}, v \cdot \nabla]\theta\|_{B^0_{\infty,r}} \leq C(\|\omega\|_{L^\infty} + \|\omega\|_{L^\rho})\big(\|\theta\|_{B^\epsilon_{\infty,r}} + \|\theta\|_{L^\rho}\big),$$

for every smooth scalar function θ.

Proof of Commutator Estimate

The following inequality will be useful to give the proof of Proposition 6.

Lemma 7

Given $(p, m) \in [1, \infty]^2$ such that $p \geq m'$ with m' the conjugate exponent of m. Let f, g and h be three functions such that $\nabla f \in L^p, g \in L^m$ and $xh \in L^{m'}$. Then,

$$\|h * (fg) - f(h * g)\|_{L^p} \leq \|xh\|_{L^{m'}} \|\nabla f\|_{L^p} \|g\|_{L^m}.$$

Proof of Commutator Estimate

We split the commutator into three parts,

$$\begin{aligned}
[\mathcal{R}, v\cdot\nabla]\theta &= \sum_{q\in\mathbb{N}}[\mathcal{R}, S_{q-1}v\cdot\nabla]\Delta_q\theta + \sum_{q\in\mathbb{N}}[\mathcal{R}, \Delta_q v\cdot\nabla]S_{q-1}\theta \\
&\quad + \sum_{q\geq -1}[\mathcal{R}, \Delta_q v\cdot\nabla]\widetilde{\Delta}_q\theta \\
&= \sum_{q\in\mathbb{N}} \mathrm{I}_q + \sum_{q\in\mathbb{N}} \mathrm{II}_q + \sum_{q\geq -1} \mathrm{III}_q \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

We start with the estimate of the first term I. According to the *iii*) of Proposition 4 there exists $h \in \mathcal{S}$ whose spectum does not meet the origin such that

$$\mathrm{I}_q(x) = h_q * (S_{q-1}v \cdot \nabla \Delta_q \theta) - S_{q-1}v \cdot (h_q * \nabla \Delta_q \theta),$$

where $h_q(x) = 2^{dq} h(2^q x)$.

Applying Lemma 7 $(\|h * (fg) - f(h * g)\|_{L^p} \leq \|xh\|_{L^{m'}} \|\nabla f\|_{L^p} \|g\|_{L^m})$. with $m = \infty$ we get

$$\begin{aligned} \|\mathrm{I}_q\|_{L^p} &\lesssim \|xh_q\|_{L^1} \|\nabla S_{q-1}v\|_{L^p} \|\Delta_q \nabla \theta\|_{L^\infty} \\ &\lesssim \|\nabla v\|_{L^p} \|\Delta_q \theta\|_{L^\infty}. \end{aligned} \tag{35}$$

In the last line we've used Bernstein inequality and $\|xh_q\|_{L^1} = 2^{-q}\|xh\|_{L^1}$.

Combined with the trivial fact

$$\Delta_j \sum_q \mathrm{I}_q = \sum_{|j-q|\le 4} \mathrm{I}_q$$

this yields

$$\begin{aligned} \|\mathrm{I}\|_{B^0_{p,r}} &\lesssim \Big(\sum_{q\ge -1} \|\mathrm{I}_q\|^r_{L^p}\Big)^{\frac{1}{r}} \\ &\lesssim \|\nabla v\|_{L^p} \|\theta\|_{B^0_{\infty,r}}. \end{aligned}$$

Let us move to the second term II. As before one writes

$$\mathrm{II}_q(x) = h_q * (\Delta_q v \cdot \nabla S_{q-1}\theta) - \Delta_q v \cdot (h_q * \nabla S_{q-1}\theta),$$

and then we obtain the estimate

$$\begin{aligned} \|\mathrm{II}_q\|_{L^p} &\lesssim 2^{-q}\|\Delta_q \nabla v\|_{L^p}\|S_{q-1}\nabla\theta\|_{L^\infty} \\ &\lesssim \|\nabla v\|_{L^p} \sum_{j\leq q-2} 2^{j-q}\|\Delta_j\theta\|_{L^\infty}. \end{aligned}$$

Combined with convolution inequalities this yields

$$\|\mathrm{II}\|_{B^0_{p,r}} \lesssim \|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,r}}.$$

Let us now deal with the third term III. Using that the divergence of $\Delta_q v$ vanishes, we rewrite III as

$$\begin{aligned} \mathrm{III} &= \sum_{q\geq 2} \mathcal{R}\mathrm{div}(\Delta_q v\, \widetilde{\Delta}_q \theta) - \sum_{q\geq 2} \mathrm{div}(\Delta_q v\, \mathcal{R}\widetilde{\Delta}_q \theta) \\ &\quad + \sum_{q\leq 1} [\mathcal{R}, \Delta_q v \cdot \nabla] \widetilde{\Delta}_q \theta \\ &= J_1 + J_2 + J_3. \end{aligned}$$

Using Proposition 4 *ii*), $(\||\mathrm{D}|^s \chi(2^{-q}|\mathrm{D}|)\mathcal{R}\|_{\mathcal{L}(L^p)} \leq C2^{qs})$ we get

$$\left\| \Delta_j \mathcal{R}\mathrm{div}(\Delta_q v\, \widetilde{\Delta}_q \theta) \right\|_{L^p} \lesssim 2^j \|\Delta_q v\|_{L^p} \|\widetilde{\Delta}_q \theta\|_{L^\infty}.$$

Also, since $\widetilde{\Delta}_q\theta$ is supported away from zero for $q \geq 2$ then Proposition 4 *iii*) $(\mathcal{R}f = 2^{qd}\psi(2^q\cdot) * f\,)$ yields

$$
\begin{aligned}
\left\|\Delta_j \mathrm{div}(\Delta_q v\, \mathcal{R}\widetilde{\Delta}_q\theta)\right\|_{L^p} &\lesssim 2^j\|\Delta_q v\|_{L^p}\|\mathcal{R}\widetilde{\Delta}_q\theta\|_{L^\infty}\\
&\lesssim 2^j\|\Delta_q v\|_{L^p}\|\widetilde{\Delta}_q\theta\|_{L^\infty}.
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
\|\Delta_j(J_1+J_2)\|_{L^p} &\lesssim \sum_{\substack{q\in\mathbb{N}\\ q\geq j-4}} 2^j\|\Delta_q v\|_{L^p}\|\widetilde{\Delta}_q\theta\|_{L^\infty}\\
&\lesssim \|\nabla v\|_{L^p}\sum_{\substack{q\in\mathbb{N}\\ q\geq j-4}} 2^{j-q}\|\Delta_q\theta\|_{L^\infty},
\end{aligned}
$$

where we have again used Bernstein inequality to get the last line. It suffices now to use convolution inequalities to get

$$
\|J_1+J_2\|_{B^0_{p,r}} \lesssim \|\nabla v\|_{L^p}\|\theta\|_{B^0_{\infty,r}}.
$$

For the last term J_3 we can write

$$\sum_{-1\leq q\leq 1}[\mathcal{R},\Delta_q v\cdot\nabla]\widetilde{\Delta}_q\theta(x)=\sum_{q\leq 1}[\mathrm{div}\ \widetilde{\chi}(\mathrm{D})\mathcal{R},\Delta_q v]\widetilde{\Delta}_q\theta(x),$$

where $\widetilde{\chi}$ belongs to $\mathcal{D}(\mathbb{R}^d)$. Proposition 4 ensures that div $\widetilde{\chi}(\mathrm{D})\mathcal{R}$ is a convolution operator with a kernel $\tilde{h}$ satisfying

$$|\tilde{h}(x)|\lesssim(1+|x|)^{-d-1}.$$

Details refer to see the proof of Proposition 3.1. Hmidi, Keraani and Rousset 2011. Thus

$$J_3=\sum_{q\leq 1}\tilde{h}*(\Delta_q v\cdot\tilde{\Delta}_q\theta)-\Delta_q v\cdot(\tilde{h}*\tilde{\Delta}_q\theta).$$

First of all we point out that $\Delta_j J_3 = 0$ for $j \geq 6$, thus we just need to estimate the low frequencies of J_3. Noticing that $x\tilde{h}$ belongs to $L^{p'}$ for $p' > 1$ then using Lemma 7 $(\|h * (fg) - f(h * g)\|_{L^p} \leq \|xh\|_{L^{m'}} \|\nabla f\|_{L^p} \|g\|_{L^m})$ with $m = p \geq 2$ we obtain

$$\begin{aligned} \|\Delta_j J_3\|_{L^p} &\lesssim \sum_{q \leq 1} \|x\tilde{h}\|_{L^{p'}} \|\Delta_q \nabla v\|_{L^p} \|\widetilde{\Delta}_q \theta\|_{L^p} \\ &\lesssim \|\nabla v\|_{L^p} \sum_{-1 \leq q \leq 1} \|\Delta_q \theta\|_{L^p}. \end{aligned}$$

This yields finally

$$\|J_3\|_{B^0_{p,r}} \lesssim \|\nabla v\|_{L^p} \|\theta\|_{L^p}.$$

This completes the proof of the first part of Proposition 6.

To estimate the terms I and II we use two facts: the first one is $\|\Delta_q \nabla u\|_{L^\infty} \approx \|\Delta_q \omega\|_{L^\infty}$ for all $q \in \mathbb{N}$. The second one is

$$\begin{aligned}
\|\nabla S_{q-1} v\|_{L^\infty} &\lesssim \|\nabla \Delta_{-1} v\|_{L^\infty} + \sum_{j=0}^{q-2} \|\Delta_j \nabla v\|_{L^\infty} \\
&\lesssim \|\omega\|_{L^\rho} + q\|\omega\|_{L^\infty}.
\end{aligned}$$

For the remainder term we do the same analysis as before except for J_3: we apply Lemma 7 with $p = \infty$ and $m = \rho$ leading to

$$\begin{aligned}
\|\Delta_j J_3\|_{L^p} &\lesssim \sum_{q \leq 1} \|x\tilde{h}\|_{L^{\rho'}} \|\Delta_q \nabla v\|_{L^\infty} \|\widetilde{\Delta}_q \theta\|_{L^\rho} \\
&\lesssim \|\nabla v\|_{L^\rho} \sum_{-1 \leq q \leq 1} \|\Delta_q \theta\|_{L^\rho} \\
&\lesssim \|\omega\|_{L^\rho} \|\theta\|_{L^\rho}.
\end{aligned}$$

This ends the proof of the Proposition. □

General critical case

Jiu, Miao Wu and Zhang obtained the global regularity for the general critical dissipation $\alpha + \beta = 1$.
Jiu, Quansen; Miao, Changxing; Wu, Jiahong; Zhang, Zhifei The two-dimensional incompressible Boussinesq equations with general critical dissipation. SIAM J. Math. Anal. 46 (2014).

Theorem 8 (Jiu, Miao, Wu, Zhang, 2014)

Let $\alpha_0 < \alpha < 1$ *and* $\alpha + \beta = 1$, *where*

$$\alpha_0 = \frac{23 - \sqrt{145}}{12} \approx 0.9132. \tag{36}$$

Assume that $u_0 \in B^{\sigma}_{2,1}(\mathbb{R}^2)$ *with* $\sigma \geq \frac{5}{2}$ *and* $\theta_0 \in B^{2}_{2,1}(\mathbb{R}^2)$. *Then the fractional Boussinesq has a unique global solution* (u, θ) *satisfying, for any* $0 < T < \infty$,

$$\begin{aligned} u &\in C([0,T]; B^{\sigma}_{2,1}(\mathbb{R}^2)) \cap L^1([0,T]; B^{\sigma+\alpha}_{2,1}(\mathbb{R}^2)), \\ \theta &\in C([0,T]; B^{2}_{2,1}(\mathbb{R}^2)) \cap L^1([0,T]; B^{2+\beta}_{2,1}(\mathbb{R}^2)). \end{aligned} \tag{37}$$

The authors also introduced the new quantity

$$G = \omega - \mathcal{R}_\alpha \theta \quad \text{with} \quad \mathcal{R}_\alpha = \Lambda^{-\alpha} \partial_1,$$

which satisfies

$$\partial_t G + u \cdot \nabla G + \Lambda^\alpha G = [\mathcal{R}_\alpha, u \cdot \nabla]\theta + \Lambda^{\beta-\alpha} \partial_1 \theta.$$

The regularity of G here does not translate to the regularity on the vorticity ω in the general case, since the corresponding regularity of $\mathcal{R}_\alpha \theta$ is not known for $\alpha < 1$.

By obtaining a suitable estimate for $[\mathcal{R}_\alpha, u\cdot\nabla]\theta$, they obtained a global bound for $\|G\|_{L^2}$ when $\alpha > \frac{4}{5}$. In addition, with $\alpha > \frac{4}{5}$, a global bound is also established for $\|G\|_{L^q}$ when q is in the range

$$2 < q < q_0 = \frac{8-4\alpha}{8-7\alpha}.$$

This global bound for $\|G\|_{L^q}$ enables them to estiamte $\|G\|_{B^s_{q,\infty}}$,

$$\|G(t)\|_{B^s_{q,\infty}} \leq C,$$

for $\alpha_0 < \alpha$ and $s \leq 3\alpha - 2$, and this will give a bound for part of ∇u.

Through the temperature equation

$$\partial_t \theta + u \cdot \nabla \theta + \Lambda^\beta \theta = 0. \tag{38}$$

They offered a different approach to gaining further regularity by θ. Since u is determined by ω through $u = \nabla^\perp \Delta^{-1} \omega$ and $\omega = G + \mathcal{R}_\alpha \theta$, So u can be decomposed into two parts,

$$u = \nabla^\perp \Delta^{-1} \omega = \nabla^\perp \Delta^{-1} G + \nabla^\perp \Delta^{-1} \mathcal{R}_\alpha \theta \equiv \widetilde{u} + v.$$

and $\widetilde{u}$ is more regular in the sense that

$$\|\nabla \widetilde{u}\|_{L^\infty} = \|\nabla \nabla^\perp \Delta^{-1} G\|_{L^\infty} \leq C \, \|G\|_{B^s_{q,\infty}} \leq C.$$

In addition, when $\alpha + \beta = 1$, v in terms of θ can be written as

$$v = \nabla^{\perp} \Delta^{-1} \Lambda^{-(1-\beta)} \partial_1 \theta.$$

Therefore, (38) is almost a generalized critical surface quasi-geostrophic (SQG) type equation.

Subcritical case

P. Constantin and V. Vicol, GFA, 2012. Global regularity for the subcritical case by a different approach.

Theorem 9 (P. Constantin and V. Vicol 2012)

Assume that $(u_0, \theta_0) \in \mathcal{S}$, the Schwartz class. If $\beta > \frac{2}{2+\alpha}$, then the Boussinesq equations has a unique global smooth solution.

The proof involves the pointwise inequality for fractional Laplacian

$$\nabla f \cdot \Lambda^\alpha \nabla f(x) \geq \frac{1}{2}\Lambda^\alpha |\nabla f|^2 + \frac{|\nabla f(x)|^{2+\alpha}}{c\,\|f\|_{L^\infty}^\alpha}.$$

Note that in this theorem $\alpha + \beta > 1$, a subcritical case.