# Theory on Well-posedness of Boussinesq Equations

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# Outline I

# 1. Background

The standard 2D Boussinesq equation can be written as

$$\begin{cases} \vec{v}_t + \vec{v} \cdot \nabla \vec{v} = -\nabla p + \nu \Delta \vec{v} + \theta \vec{e}_2, \\ \nabla \cdot \vec{v} = 0, \\ \theta_t + \vec{v} \cdot \nabla \theta = \kappa \Delta \theta, \end{cases}$$

- $\vec{v}(t,x)$: the 2D velocity field
- $p(t,x)$: the pressure
- $\theta(t,x)$: the temperature in the content of thermal convection and the density in the modeling of geophysical fluids
- $\nu$: the viscosity
- $\kappa$: the thermal diffusivity
- $\vec{e}_2 = (0,\, 1)$ is the unit vector in the vertical direction.

The Boussinesq equations model large scale atmospheric and oceanic flows that are responsible for cold fronts and the jet stream.

• A.E. Gill, Atmosphere-Ocean Dynamics, Academic Press (London), 1982.

• J. Pedlosky, Geophysical Fluid Dyanmics, Springer-Verlag, New York, 1987.

In addition, the Boussinesq equations also play an important in the study of Rayleigh-Benard convection.

• P. Constantin and C.R. Doering Infinite Prandtl number convection, J. Statistical Physics 94 (1999).

Mathematically the 2D Boussinesq equations serve as a lower dimensional model of the 3D hydrodynamics equations. In fact, the 2D Boussinesq equations retain some key features of the 3D Euler and Navier-Stokes equations such as the vortex stretching mechanism. The inviscid 2D Boussinesq equations are identical to the Euler equations for the 3D axisymmetric swirling flows.

- A.J. Majda and A.L. Bertozzi, Vorticity and Incompressible Flow, Cambridge University Press, 2001.

3D rotating Boussinesq equations $\Longrightarrow$

1. primitive equations
2. 2D Boussinesq equations
3. surface quasi-geostrophic (SQG) equation

Fluid flows in atmosphere and ocean have two distinctive features: rotation and stratification. The simplest model that contains both features is the 3D rotating Boussinesq equation:

$$\begin{cases} \partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} + f\,\vec{e_3} \times \vec{u} = \nu \Delta \vec{u} - \frac{1}{\rho_b} \nabla p + \rho g\, \vec{e_3}, \\ \nabla \cdot \vec{u} = 0, \\ \partial_t \rho + \vec{u} \cdot \nabla \rho = \kappa \Delta \rho, \end{cases} \tag{1.1}$$

where $f = 2\Omega \sin\phi$ with $\Omega$ being the angular frequency of planetary rotation and $\phi$ the latitude, $\rho_b$ is a constant for reference density, $\vec{e_3} = (0, 0, 1)$, $f\,\vec{e_3} \times \vec{u}$ represents the Coriolis forcing.

More explicitly, if $\vec{u} = (u, v, w)$, then

$$f\vec{e_3} \times \vec{u} = f \begin{pmatrix} -v \\ u \\ 0 \end{pmatrix}$$

and (1.1) becomes

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + w\partial_z u - fv = \nu\Delta u - \frac{1}{\rho_b}\partial_x p, \\ \partial_t v + u\partial_x v + v\partial_y v + w\partial_z v + fu = \nu\Delta v - \frac{1}{\rho_b}\partial_y p, \\ \partial_t w + u\partial_x w + v\partial_y w + w\partial_z w = \nu\Delta w - \frac{1}{\rho_b}\partial_z p + \rho g, \\ \partial_x u + \partial_y v + \partial_z w = 0, \\ \partial_t \rho + u\partial_x \rho + v\partial_y \rho + w\partial_z \rho = \kappa\Delta\rho. \end{cases} \tag{1.2}$$

For atmospheric and oceanic flows in the mid-latitude, the $w-$equation can be simplified. The terms involving $w$ in the $w-$equation are small and the $w$-equation is reduced to

$$\frac{1}{\rho_b}\partial_z p - \rho g = 0. \tag{1.3}$$

This is the so called the hydrostatic balance. (1.3) provides a special solution of (1.2). In fact, $\vec{u} = 0$ with $p$ and $\rho$ satisfying (1.3) solves (1.2). The system of equations containing the $u$-equation, the $v$-equation in (1.2) and

$$\begin{aligned}
&\frac{1}{\rho_b}\partial_z p - \rho g = 0, \\
&\partial_x u + \partial_y v + \partial_z w = 0, \\
&\partial_t \rho + u\partial_x \rho + v\partial_y \rho + w\partial_z \rho = \kappa \Delta \rho
\end{aligned}$$

are called the primitive equations.

If $f \equiv 0$, (1.2) becomes the 3D Boussinesq equations without rotation. If $f \equiv 0$ and all physical quantities are independent of $z$, then (1.2) reduces to the 2D Boussinesq equations, which read

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u = -\partial_x p + \nu\Delta u, \\ \partial_t v + u\partial_x v + v\partial_y v = -\partial_y p + \nu\Delta v + \theta, \\ \partial_x u + \partial_y v = 0, \\ \partial_t \theta + u\partial_x \theta + v\partial_y \theta = \kappa\Delta\theta. \end{cases}$$

Under some circumstances, the 3D rotation Boussinesq reduces to the surface quasi-geostrophic (SQG) equation. The Rossby number indicates the ratio of the inertial to the strength of rotation. In low-pressure systems, the Rossby number is small and the balance is between Coriolis and pressure forces, namely

$$f\,\vec{e_3} \times \vec{u} = -\frac{1}{\rho_b}\nabla p.$$

This is the so-called geostrophic balance. In terms of their components,

$$f\begin{pmatrix} -v \\ u \end{pmatrix} = -\frac{1}{\rho_b}\begin{pmatrix} \partial_x p \\ \partial_y p \end{pmatrix}$$

or simply $f\,\rho_b\,\vec{u}_H = \nabla^{\perp} p$. After ignoring the dissipation and removing the geostrophic balance from (1.2),

$$\partial_t u + u\partial_x u + v\partial_y u = 0,$$
$$\partial_t v + u\partial_x v + v\partial_y v = 0.$$

Then, $\omega = \partial_x v - \partial_y u$ satisfies

$$\partial_t \omega + u\partial_x \omega + v\partial_y \omega = 0, \quad \frac{D\omega}{Dt} = 0. \tag{1.4}$$

In terms of the stream function $\psi$,

$$\omega = \Delta\psi = \partial_x^2 \psi + \partial_y^2 \psi + \frac{\partial}{\partial z}\left(\left(\frac{f}{N}\right)^2 \frac{d\psi}{dz}\right)$$

where $N = \sqrt{-g\partial_z \bar{\rho}}$ denotes the the buoyancy frequency. (1.4) indicates that, if $\omega$ is a constant initially, it remains a constant. Suppose that $\omega_0 = 0$ and re-scaling the $z-$component, we find that

$$\Delta\psi = 0.$$

In addition, for small Rossby number and through re-scaling,

$$\rho = \frac{\partial \psi}{\partial z}$$

and

$$\partial_t \rho + u \partial_x \rho + v \partial_y \rho = 0 \quad \text{or} \quad \frac{D\rho}{Dt} = 0.$$

**Lemma**

*If $g$ is a bounded smooth function in $\mathbb{R}^d$, then*

$$\begin{cases} \Delta \psi = 0 & \mathbb{R}^d \times \mathbb{R}^+ \\ \psi = g & \mathbb{R}^d \end{cases}$$

*has a bounded smooth solution $\psi$, and*

$$\left.\frac{\partial \psi}{\partial z}\right|_{z=0} = (-\Delta)^{\frac{1}{2}} g \quad \text{on} \quad \mathbb{R}^d$$

Applying above lemma and writing $\theta$ for $\rho$, we obtain

$$\begin{cases} \partial_t \theta + u \partial_x \theta + v \partial_y \theta = 0 \\ u = \nabla^\perp \psi, \quad (-\Delta)^{\frac{1}{2}} \psi = \theta, \end{cases}$$

which is the SQG equation.

# 2.1
# Local existence and blow-up criterion

# Local existence and blow-up criterion

In this section, we introduce the local existence and uniqueness of smooth solutions of the inviscid Boussinesq equations by Chae Dongho. They also obtain a blow-up criterion for these smooth solutions.

- Chae, Dongho; Nam, Hee-Seok, Local existence and blow-up criterion for the Boussinesq equations. Proc. Roy. Soc. Edinburgh Sect. A 127 (1997), no. 5, 935–946.

# Sobolev space

$$W^{k,p}(\Omega) := \{u(x) \in L^p(\Omega) \,|\, D^\alpha u \in L^P(\Omega),\ 0 \leq |\alpha| \leq k\}$$

with the standard norm given by

$$\|u\|^p_{W^{k,p}(\Omega)} := \sum_{0 \leq |\alpha| \leq k} \|\partial^\alpha u\|^p_{L^p(\Omega)}, \quad 1 \leq p < \infty;$$

$$\|u\|_{W^{k,\infty}(\Omega)} := \sum_{0 \leq |\alpha| \leq k} \|\partial^\alpha u\|_{L^\infty(\Omega)}.$$

Denote $H^m = W^{m,2}$ .

Introduce the function space

$$V^m(\mathbb{R}^2) = \{v \in H^m(\mathbb{R}^2); \nabla \cdot v = 0\}.$$

Set $\Lambda = (-\Delta)^{\frac{1}{2}}$.

## Product estimates

$$\|\Lambda^s(uv)\|_{L^p} \leq C(\|u\|_{L^{p_1}}\|\Lambda^s v\|_{L^{p_2}} + \|v\|_{L^{p_3}}\|\Lambda^s u\|_{L^{P_4}}),$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$, $p,\ p_2,\ p_4 \in (1,\ \infty)$.

## Commutator Estimate $(1 < p < \infty,\ s \geq 0)$

$$\|\Lambda^s(u\,v) - u\,\Lambda^s v\|_{L^p} \leq\ C(\|\nabla u\|_{L^{p_1}}\|\Lambda^{s-1}v\|_{L^{p_2}} + \|v\|_{L^{p_3}}\|\Lambda^s u\|_{L^{p_4}}),$$

where$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$, $p,\ p_2,\ p_4 \in (1,\ \infty)$.

## Gagliado-Nirenberg inequality

$$\|\Lambda^s u\|_{L^p} \leq C\|\Lambda^{s_1}u\|_{L^{p_1}}^{\theta}\|\Lambda^{s_2}u\|_{L^{p_2}}^{1-\theta},$$

where $0 < \theta < 1,\ 1 \leq p,\ p_1,\ p_2 \leq \infty$

$$\frac{n}{p} - s = \theta\left(\frac{n}{p_1} - s_1\right) + (1-\theta)\left(\frac{n}{p_2} - s_2\right).$$

## Lemma (The Hodge Decomposition in $H^m$)

*Every vector field $v \in H^m\left(\mathbb{R}^d\right)$, $m \in \mathbb{Z}^+ \cup \{0\}$, has the unique orthogonal decomposition*

$$v = w + \nabla\varphi,$$

*such that the Leray's projection operator $\mathcal{P}v = w$ on the divergence-free functions satisfies*
*(i) $\mathcal{P}v, \nabla\varphi \in H^m$, $\int_{\mathbb{R}^N} \mathcal{P}v \cdot \nabla\varphi dx = 0$, $div\mathcal{P}v = 0$, and*

$$\|\mathcal{P}v\|_m^2 + \|\nabla\varphi\|_m^2 = \|v\|_m^2,$$

*(ii) $\mathcal{P}$ commutes with the distribution derivatives,*

$$\mathcal{P}D^\alpha v = D^\alpha \mathcal{P}v, \quad \forall v \in H^m, \quad |\alpha| \leq m,$$

## Lemma 2

$(iii)$ $\mathcal{P}$ commutes with mollifiers $\mathcal{J}_\epsilon$,

$$\mathcal{P}\left(\mathcal{J}_\epsilon v\right) = \mathcal{J}_\epsilon(\mathcal{P}v), \quad \forall v \in H^m, \quad \epsilon > 0.$$

$(iv)$ $\mathcal{P}$ is symmetric,

$$(\mathcal{P}u, v)_m = (u, \mathcal{P}v)_m.$$

(Details see p71,99 of A.J. Majda and A.L. Bertozzi, Vorticity and Incompressible Flow.)

For 2D inviscid Boussinesq eqautions with a external potential force $f(t,x)$(i.e. $\operatorname{curl} f = 0$)

$$\begin{cases} v_t + v\cdot\nabla v = -\nabla p + \theta f, & t\in\mathbb{R}_+,\ x\in\mathbb{R}^2, \\ \theta_t + v\cdot\nabla\theta = 0, \\ \nabla\cdot v = 0, \\ v\big|_{t=0} = v_0, \quad \theta\big|_{t=0} = \theta_0. \end{cases} \tag{2.1}$$

**Theorem 1 (Chae, Dongho; Nam, Hee-Seok 1997)**

Let the initial data $(v_0,\theta_0)\in V^m(\mathbb{R}^2)\times H^m(\mathbb{R}^2)$ and suppose that $f\in L^\infty_{\mathrm{loc}}([0,\infty); W^{m,\infty}(\mathbb{R}^2))$ for some $m>2$. Then there exists the unique solution $(v,\theta)\in C([0,T]; V^m(\mathbb{R}^2)\times H^m(\mathbb{R}^2))$ to the Boussinesq equations (2.1).

# Mollifier

Mollifier: The function satisfies the following conditions,

$$\rho(x) \in C_0^\infty(\mathbb{R}^2), \quad \rho(x) \geq 0, \quad \int_{\mathbb{R}^2} \rho(x)dx = 1.$$

The mollification $J_\epsilon v$ of functions $v \in L^p(\mathbb{R}^2)$ for $1 \leq p \leq \infty$ and $\epsilon > 0$

$$(J_\epsilon v)(x) = \epsilon^{-2} \int_{\mathbb{R}^2} \rho\left(\frac{x-y}{\epsilon}\right) v(y)dy, \qquad \epsilon > 0.$$

## Lemma 1

Let $J_\epsilon$ be the mollifier defined as above. Then:

(i) $\|J_\epsilon v\|_{C^0} \leq \|v\|_{C^0}, \ \forall v \in C^0\left(\mathbb{R}^2\right)$;

(ii) $D^\alpha J_\epsilon v = J_\epsilon D^\alpha v, \ \forall |\alpha| \leq m, \ v \in H^m\left(\mathbb{R}^2\right)$;

(iii) $\lim_{\epsilon \searrow 0} \|J_\epsilon v - v\|_{H^m} = 0, \ \forall v \in H^m\left(\mathbb{R}^2\right)$;

(iv) for all $v \in H^m\left(\mathbb{R}^2\right)$, $k \in \mathbb{N} \cup \{0\}$, $\epsilon > 0$,

$$\|J_\epsilon v\|_{H^{m+k}} \leq \frac{c_{m,k}}{\epsilon^k}\|v\|_{H^m} \quad \text{and} \quad \|J_\epsilon D^k v\|_{C^0} \leq \frac{c_k}{\epsilon^{1+k}}\|v\|_{L^2};$$

(v) $\int_{\mathbb{R}^2} (J_\epsilon u) v dx = \int_{\mathbb{R}^2} u (J_\epsilon v) dx, \ \forall u \in L^p\left(\mathbb{R}^2\right), v \in L^q\left(\mathbb{R}^2\right),$
$(1/p) + (1/q) = 1.$

(Details see p131 of A.J. Majda and A.L. Bertozzi, Vorticity and Incompressible Flow.)

Now, we regularise the Boussinesq equations (2.1) as follows:

$$\begin{aligned} v_t^\epsilon + J_\epsilon\left(\left(J_\epsilon \epsilon^\epsilon\right)\cdot\nabla\left(J_\epsilon v^\epsilon\right)\right) &= -\nabla p^\epsilon + \theta^\epsilon f, \\ \theta_t^\epsilon + J_\epsilon\left(\left(J_\epsilon v^\epsilon\right)\cdot\nabla\left(J_\epsilon \theta^\epsilon\right)\right) &= 0, \\ \operatorname{div} v^\epsilon &= 0. \end{aligned} \tag{2.2}$$

Projecting (2.2) onto $V^m\left(R^2\right)$, we eliminate $\nabla p^\epsilon$ and the incompressibility condition div $v^\epsilon = 0$.

$$\begin{aligned} v_t^\epsilon &= -PJ_\epsilon\left(\left(J_\epsilon v^\epsilon\right)\cdot\nabla\left(J_\epsilon v^\epsilon\right)\right) + P\left(\theta^\epsilon f\right), \\ \theta_t^\epsilon &= -J_\epsilon\left(\left(J_\epsilon v^\epsilon\right)\cdot\nabla\left(J_\epsilon \theta^\epsilon\right)\right), \\ v^\epsilon\big|_{t=0} = v_0, &\quad \theta^\epsilon\big|_{t=0} = \theta_0. \end{aligned} \tag{2.3}$$

We consider

$$\tilde{v}^{\epsilon} = \begin{pmatrix} v^{\epsilon} \\ \theta^{\epsilon} \end{pmatrix} \in V^{m}\left(\mathbb{R}^{2}\right) \times H^{m}\left(\mathbb{R}^{2}\right)$$

with the norm

$$\left\|\tilde{v}^{\epsilon}\right\|_{m} := \left\|v^{\epsilon}\right\|_{V^{m}} + \left\|\theta^{\epsilon}\right\|_{H^{m}}.$$

Thus we reduce the regularised Boussinesq equations (2.3) to

$$\begin{cases} \dfrac{d}{dt}\tilde{v}^{\epsilon} = F_{\epsilon}\left(\tilde{v}^{\epsilon}\right), \\ \tilde{v}^{\epsilon}\big|_{t=0} = \tilde{v}_{0}^{\epsilon} = \begin{pmatrix} v_{0} \\ \theta_{0} \end{pmatrix}, \end{cases} \tag{2.4}$$

where

$$F_\epsilon(\tilde{v}^\epsilon) = F_\epsilon \begin{pmatrix} v^\epsilon \\ \theta^\epsilon \end{pmatrix} = \begin{pmatrix} F_\epsilon^1(\tilde{v}^\epsilon) \\ F_\epsilon^2(\tilde{v}^\epsilon) \end{pmatrix}$$
$$= \begin{pmatrix} -PJ_\epsilon\left(\left(J_\epsilon v^\epsilon\right)\cdot\nabla\left(J_\epsilon v^\epsilon\right)\right) + P\left(\theta^\epsilon f\right) \\ -J_\epsilon\left(\left(J_\epsilon v^\epsilon\right)\cdot\nabla\left(J_\epsilon \theta^\epsilon\right)\right) \end{pmatrix}.$$

By Picard's theorem for ODEs in a Banach space, we can prove local existence and uniqueness of solutions $\tilde{v}^\epsilon$.

## Lemma 2

Let $F_\epsilon : V^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \to V^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2)$ be defined as above for some $m > 1$. Suppose that $f \in W^{m,\infty}(\mathbb{R}^2)$. Then $F_\epsilon$ is locally Lipschitz continuous.

Proof. First we prove

$$F_\epsilon : V^m\left(R^2\right) \times H^m\left(R^2\right) \to V^m\left(R^2\right) \times H^m\left(R^2\right)$$

Since $\operatorname{div} v^\epsilon = 0$, and $\operatorname{div} Pu = 0$, $\forall u \in H^m\left(\mathbb{R}^2\right)$, we get

$$F_\epsilon^1\left(\tilde{v}^\epsilon\right) \in V^m\left(\mathbb{R}^2\right) \text{ and } F_\epsilon^2\left(\tilde{v}^\epsilon\right) \in H^m\left(\mathbb{R}^2\right).$$

Let

$$\begin{pmatrix} F_\epsilon^1(\tilde{v}^\epsilon) \\ F_\epsilon^2(\tilde{v}^\epsilon) \end{pmatrix} = \begin{pmatrix} -PJ_\epsilon\left(\left(J_\epsilon v^\epsilon\right)\cdot\nabla\left(J_\epsilon v^\epsilon\right)\right) + P\left(\theta^\epsilon f\right) \\ -J_\epsilon\left(\left(J_\epsilon v^\epsilon\right)\cdot\nabla\left(J_\epsilon \theta^\epsilon\right)\right) \end{pmatrix}$$
$$=: \begin{pmatrix} F_\epsilon^{11}(\tilde{v}^\epsilon) + F_\epsilon^{12}(\tilde{v}^\epsilon) \\ F_\epsilon^{2}(\tilde{v}^\epsilon) \end{pmatrix}.$$

For $F_\epsilon^{11}$

$$\begin{aligned} \left\|F_\epsilon^{11}(\tilde{v}_1^\epsilon) - F_\epsilon^{11}(\widetilde{v_2^\epsilon})\right\|_{V^m} \leq & \left\|PJ_\epsilon\left(\left(J_\epsilon v_1^\epsilon\right)\cdot\nabla\left(J_\epsilon\left(v_1^\epsilon - v_2^\epsilon\right)\right)\right)\right\|_{H^m} \\ & + \left\|PJ_\epsilon((J_\epsilon(v_1^\epsilon - v_2^\epsilon))\cdot\nabla J_\epsilon v_2^\epsilon)\right\|_{H^m} \\ =: & \, I + II. \end{aligned}$$

Since $m > 1$, the commutator-type estimate (see p129 of A.J. Majda and A.L. Bertozzi, Vorticity and Incompressible Flow.) provides us with

$$\begin{aligned}
I &= \|PJ_\varepsilon\left(\left(J_\varepsilon v_1^\varepsilon\right)\cdot\nabla\left(J_\varepsilon\left(v_1^\varepsilon - v_2^\varepsilon\right)\right)\right)\|_{H^m} \\
&\leq C\|\left(J_\varepsilon v_1^\varepsilon\right)\cdot\nabla\left(J_\varepsilon\left(v_1^\varepsilon - v_2^\varepsilon\right)\right)\|_{H^m} \\
&\leq C\{\|J_\varepsilon v_1^\varepsilon\|_{L^\infty}\|\nabla J_\varepsilon\left(v_1^\varepsilon - v_2^\varepsilon\right)\|_{H^m} + \|J_\varepsilon v_1^\varepsilon\|_{H^m}\|\nabla J_\varepsilon\left(v_1^\varepsilon - v_2^\varepsilon\right)\|_{L^\infty}\} \\
&\leq \frac{C}{\varepsilon}\|v_1^\varepsilon\|_{H^m}\|v_1^\varepsilon - v_2^\varepsilon\|_{H^m}.
\end{aligned}$$

Here we used the Lemma 1 (iv) in the last inequality. Similarly, we obtain

$$II \leq \frac{C}{\varepsilon}\|v_2^\varepsilon\|_{H^m}\|v_1^\varepsilon - v_2^\varepsilon\|_{H^m}.$$

On the other hand, since $m > 1$,

$$\begin{aligned}\left\|F_{\varepsilon}^{12}(\tilde{v}_1^{\varepsilon}) - F_{\varepsilon}^{12}(\tilde{v}_2^{\epsilon})\right\|_{H^m} &= \|P(\theta_1^{\varepsilon} f) - P(\theta_2^{\varepsilon} f)\|_{H^m} \\ &\leq \|\theta_1^{\varepsilon} - \theta_2^{\varepsilon}\|_{H^m} \|f\|_{W^{m,\infty}}.\end{aligned}$$

Combining the above estimates, we have

$$\|F_{\varepsilon}^{1}(\tilde{v}_1^{\varepsilon}) - F_{\varepsilon}^{1}(\tilde{v}_2^{\epsilon})\|_m \leq C(\epsilon, \|f\|_{W^{m,\infty}}, \|\tilde{v}_1^{\epsilon}\|_{H^m}, \|\tilde{v}_2^{\epsilon}\|_{H^m}) \|\tilde{v}_1^{\epsilon} - \tilde{v}_2^{\epsilon}\|_m.$$

For $F_{\varepsilon}^{2}(\tilde{v}^{\varepsilon})$, which estimate is similar with $F_{\epsilon}^{11}$, we obtain

$$\|F_{\varepsilon}^{2}(\tilde{v}_1^{\varepsilon}) - F_{\varepsilon}^{2}(\tilde{v}_2^{\epsilon})\|_m \leq \frac{C}{\epsilon} \|\tilde{v}_1^{\epsilon}\|_{H^m} \|\theta_1^{\epsilon} - \theta_2^{\epsilon}\|_{H^m} \leq C(\epsilon, \|\tilde{v}_1^{\epsilon}\|_{H^m}) \|\tilde{v}_1^{\epsilon} - \tilde{v}_2^{\epsilon}\|_m.$$

Finally, we have

$$\|F_\varepsilon(\tilde{v}_1^\varepsilon) - F_\varepsilon(\tilde{v}_2^\epsilon)\|_m \leq C(\epsilon, \|f\|_{W^{m,\infty}}, \|\tilde{v}_1^\epsilon\|_{H^m}, \|\tilde{v}_2^\epsilon\|_{H^m})\|\tilde{v}_1^\epsilon - \tilde{v}_2^\epsilon\|_m.$$

□

The following is a corollary of Picard's theorem and the above Lemma.

## Proposition 1

Let $\tilde{v}_0 \in V^m(\mathbb{R}^2) \times H^m(R^2)$ for $m > 1$ be given and suppose that $f \in L^\infty([0, T]; W^{m,\infty}(\mathbb{R}^2))$. Then, for any given $\varepsilon > 0$, there exists the unique solution $\tilde{v}^\epsilon \in C^1([0, T_\varepsilon); V^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2))$ for some $T_\varepsilon = T_\varepsilon(\|\tilde{v}_0\|_m) > 0$.

Next, we continue the local solution of the regularised problem globally in time.

## Proposition 2

Under the same assumptions as in Proposition 1, for any $\epsilon > 0$, the regularised solution $\tilde{v}^\epsilon$ exists globally in time,

$$\tilde{v}^\epsilon \in C^1\left([0,\infty); V^m\left(\mathbb{R}^2\right) \times H^m\left(\mathbb{R}^2\right)\right).$$

Proof. First, we prove the $L^2$-energy estimate. we take the $L^2$-inner product of the equation (2.3) with $v^\varepsilon$ and $\theta^\varepsilon$, respectively, and integrate by parts, so that we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|v^\varepsilon(\cdot,t)\right\|_{L^2}^2 &\le \left\|P\left(\theta^\varepsilon f\right)v^\varepsilon(\cdot,t)\right\|_{L^1}\\ &\le \left\|f(\cdot,t)\right\|_{L^\infty}\left\|\theta^\varepsilon(\cdot,t)\right\|_{L^2}\left\|v^\varepsilon(\cdot,t)\right\|_{L^2},\end{aligned}$$

and

$$\frac{1}{2}\frac{d}{dt}\|\theta^{\varepsilon}(\cdot,t)\|_{L^2}^2 = 0.$$

Thus we get $\|\theta^{\varepsilon}(\cdot,t)\|_{L^2} = \|\theta_0\|_{L^2}$ for all $t \geqq 0$, and

$$\frac{d}{dt}\|\tilde{v}^{\epsilon}(\cdot,t)\|_0 \leq \|\theta_0\|_{L^2}\|f(\cdot,t)\|_{L^\infty}.$$

Integrating over $[0,t]$, we obtain

$$\sup_{0\leq t\leq T}\|\tilde{v}^{\epsilon}(\cdot,t)\|_0 \leq \|\tilde{v}_0\|_0\left(1+\sup_{0\leq t\leq T}\|f(\cdot,t)\|_{L^\infty}T\right).$$

Next, we prove the $H^m$-energy estimates. Let us consider the first equation in (2.3),

$$v_t^\varepsilon = -PJ_\varepsilon\left(\left(J_\varepsilon v^\varepsilon\right)\cdot\nabla\left(J_\varepsilon v^\varepsilon\right)\right) + P\left(\theta^\varepsilon f\right).$$

We apply the operator $D^\alpha$ on each side of the equation, multiply the result by $D^\alpha v^\varepsilon$, integrate over $\mathbb{R}^2$ and sum over $|\alpha|\leq m$. Since $\mathrm{div} v^\varepsilon = 0$, by calculus inequality, we then have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v^\varepsilon\|_{H^m}^2 \\
=&-\sum_{|\alpha|\le m}\left\{\int_{\mathbb{R}^2}\left(D^\alpha PJ_\varepsilon\left(\left(J_\varepsilon v^\varepsilon\right)\cdot\nabla\left(J_\varepsilon v^\varepsilon\right)\right)\right)D^\alpha v^\varepsilon dx+\int_{\mathbb{R}^2}\left(D^\alpha P\left(\theta^\varepsilon f\right)\right)D^\alpha v^\varepsilon dx\right\} \\
=&-\sum_{|\alpha|\le m}\left\{\int_{\mathbb{R}^2}P(D^\alpha((J_\varepsilon v^\varepsilon)\cdot\nabla\left(J_\varepsilon v^\varepsilon\right))-((J_\varepsilon v^\varepsilon)\cdot\nabla D^\alpha\left(J_\varepsilon v^\varepsilon\right)))D^\alpha J_\varepsilon v^\varepsilon dx\right. \\
&\left.+\int_{\mathbb{R}^2}\left(D^\alpha P\left(\theta^\varepsilon f\right)\right)D^\alpha v^\varepsilon dx\right\} \\
\le& C\left\{\|\nabla J_\varepsilon v^\varepsilon\|_{L^\infty}\|v^\varepsilon\|_{H^m}+\|v^\varepsilon\|_{H^m}\|\nabla J_\varepsilon v^\varepsilon\|_{L^\infty}+\|\theta^\varepsilon\|_{H^m}\|f\|_{W^{m,\infty}}\right\}\|v^\varepsilon\|_{H^m}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\frac{d}{dt}\|v^\varepsilon(\cdot,t)\|_{H^m}&\le C\left(\|\nabla J_\varepsilon v^\varepsilon(\cdot,t)\|_{L^\infty}\|v^\varepsilon(\cdot,t)\|_{H^m}+\|\theta^\varepsilon(\cdot,t)\|_{H^m}\|f(\cdot,t)\|_{W^{m,\infty}}\right) \\
&\le C\left(\|f(\cdot,t)\|_{W^{m,\infty}}+\|\nabla J_\varepsilon\tilde{v}^\varepsilon(\cdot,t)\|_{L^\infty}\right)\|\tilde{v}^\varepsilon(\cdot,t)\|_m.
\end{aligned}
$$

Using a similar process to the second equation in (2.3) leads to

$$\begin{aligned}\frac{d}{dt}\|\theta^\varepsilon(\cdot,t)\|_{H^m} &\le C\,\|\nabla J_\varepsilon \tilde{v}^\varepsilon(\cdot,t)\|_{L^\infty}\left(\|v^\varepsilon(\cdot,t)\|_{H^m}+\|\theta^\varepsilon(\cdot,t)\|_{H^m}\right)\\ &= C\,\|\nabla J_\varepsilon \tilde{v}^\varepsilon(\cdot,t)\|_{L^\infty}\,\|\tilde{v}^\varepsilon(\cdot,t)\|_m\,.\end{aligned}$$

Combining the above estimates, we get

$$\frac{d}{dt}\|\tilde{v}^\varepsilon(\cdot,t)\|_m \le C\left(\|f(\cdot,t)\|_{W^{m,\infty}}+\|\nabla J_\varepsilon \tilde{v}^\varepsilon(\cdot,t)\|_{L^\infty}\right)\|\tilde{v}^\varepsilon(\cdot,t)\|_m\,,$$

where $C$ is a constant independent of $\varepsilon > 0$. From the above two energy estimates, we have

$$\begin{aligned}&\frac{d}{dt}\|\tilde{v}^\varepsilon(\cdot,t)\|_m\\ &\le C\left(\|f(\cdot,t)\|_{W^{m,\infty}}+\|\nabla J_\varepsilon \tilde{v}^\varepsilon(\cdot,t)\|_{L^\infty}\right)\|\tilde{v}^\varepsilon(\cdot,t)\|_m\\ &\le C(\varepsilon)\left(\|f(\cdot,t)\|_{W^{m,\infty}}+\|\tilde{v}^\varepsilon(\cdot,t)\|_0\right)\|\tilde{v}^\varepsilon(\cdot,t)\|_m\\ &\le C(\varepsilon)\{\|f(\cdot,t)\|_{W^{m,\infty}}+(1+\sup_{0\le t\le T}\|f(\cdot,t)\|_{L^\infty}T)\,\|\tilde{v}_0\|_0+\|\theta_0\|_{L^2}\}\,\|\tilde{v}^\varepsilon(\cdot,t)\|_m\\ &\le C(\varepsilon,\sup_{0\le t\le T}\|f(\cdot,t)\|_{W^{m,\infty}},\|\tilde{v}_0\|_0\,,T)\,\|\tilde{v}^\varepsilon(\cdot,t)\|_m\,.\end{aligned}$$

Then Gronwall's inequality gives

$$\sup_{0\le t\le T} \|\tilde{v}^{\epsilon}(\cdot,t)\|_m \le \|\tilde{v}_0\|_m \exp[C(\varepsilon, \sup_{0\le t\le T} \|f(\cdot,t)\|_{W^{m,\infty}}, \|\tilde{v}_0\|_0, T)].$$

Thus by the standard continuation principle for ordinary differential equations, We obtain the global existence. $\square$

Remark: A similar argument to the original Boussinesq equations (2.1) gives the following $H^m$-energy estimate:

$$\frac{d}{dt}\|\tilde{v}(\cdot,t)\|_m \le C\left(\|f(\cdot,t)\|_{W^{m,\infty}} + \|\nabla\tilde{v}(\cdot,t)\|_{L^\infty}\right)\|\tilde{v}(\cdot,t)\|_m, \tag{2.5}$$

which will be used to prove the blow-up criterion of Boussinesq equations.

In order to get the existence of smooth solutions locally in time for the original Boussinesq equations (2.1), we need the following uniform estimates of $\tilde{v}^{\epsilon}$.

## Lemma 3

Let the initial condition $\tilde{v}_0 \in V^m\left(\mathbb{R}^2\right) \times H^m\left(\mathbb{R}^2\right)$, $f \in L^{\infty}_{\mathrm{loc}}([0,\infty); W^{m,\infty}\left(\mathbb{R}^2\right))$ for some $m > 2$. Then:
(i) $(\tilde{v}^{\varepsilon})$ is uniformly bounded in $C\left([0,t]; V^m\left(\mathbb{R}^2\right) \times H^m\left(\mathbb{R}^2\right)\right)$ for some time $T$ with the rough upper bound

$$T = \frac{1}{2C(1 + \sup\limits_{0 \le t \le T_0} \|f(\cdot,t)\|_{W^{m,\infty}})\,(1 + \|\tilde{v}_0\|_m)};$$

(ii) $(d/dt \;\; \tilde{v}^{\epsilon})$ is uniformly bounded in $C([0,T]; V^{m-1}(\mathbb{R}^2) \times H^{m-1}(\mathbb{R}^2))$ for the above time $T$.

Proof. From the inequality (2.5), since $m > 2$,

$$\frac{d}{dt}\|\tilde{v}^{\epsilon}(\cdot,t)\|_m \leq C\left(\|f(\cdot,t)\|_{W^{m,\infty}} + \|\tilde{v}^{\epsilon}(\cdot,t)\|_m\right)\|\tilde{v}^{\epsilon}(\cdot,t)\|_m$$
$$\leq C\left(1+\|f(\cdot,t)\|_{W^{m,\infty}}\right)\left(1+\|\tilde{v}^{\epsilon}(\cdot,t)\|_m\right)^2.$$

Using the generalised Gronwall's inequality, we have

$$1+\|\tilde{v}^{\epsilon}(\cdot,t)\|_m \leq \frac{(c_m+1)\left(1+\|\tilde{v}_0\|_m\right)}{1-C\left(1+\|\tilde{v}_0\|_m\right)\left(1+\sup_{0\leq t\leq T}\|f(\cdot,t)\|_{W^{m,\infty}}\right)t},$$

This says that the family $(\tilde{v}^{\epsilon})$ is uniformly bounded in $H^m$, $m > 2$.

$$\sup_{0\leq t\leq T}\|\tilde{v}^{\epsilon}(\cdot,t)\|_m \leq 2\left(c_m+1\right)\left(1+\|\tilde{v}_0\|_m\right)$$

for

$$T = \frac{1}{2C\left(1+\|\tilde{v}_0\|_m\right)\left(1+\sup_{0\leq t\leq T_0}\|f(\cdot,t)\|_{W^{m,\infty}}\right)}$$

This proves (i).

From the equation (2.4), we get

$$\begin{aligned}\left\|\frac{d}{dt}\tilde{v}^{\epsilon}\right\|_{m-1} &\leq \left\|PJ_{\varepsilon}\left(\left(J_{\varepsilon}v^{\varepsilon}\right)\cdot\nabla\left(J_{\varepsilon}v^{\epsilon}\right)\right)\right\|_{H^{m-1}} + \left\|P\left(\theta^{\varepsilon}f\right)\right\|_{H^{m-1}} \\ &\quad + \left\|J_{\varepsilon}\left(\left(J_{\varepsilon}v^{\varepsilon}\right)\cdot\nabla\left(J_{\varepsilon}\theta^{\varepsilon}\right)\right)\right\|_{H^{m-1}} \\ &\leq \left(1+\|f\|_{W^{m-1,\infty}}\right)\|\tilde{v}^{\epsilon}\|_{m}^{2}.\end{aligned}$$

This gives the uniform boundedness of $(d/dt\ \tilde{v}^{\epsilon})$ in $V^{m-1}(\mathbb{R}^2)\times H^{m-1}(\mathbb{R}^2)$ and the proof is complete. □

# Proof of Theorem 1.

Applying the Arzela-Ascoli theorem to the results of Lemma 3, we know that the family $(\tilde{v}^{\epsilon})$ is precompact in $C([0,T]; V^{m-1}_{\mathrm{loc}}(\mathbb{R}^2) \times H^{m-1}_{\mathrm{loc}}(\mathbb{R}^2))$(which is also precompact in $C([0,T]; V^{s}_{\mathrm{loc}}(\mathbb{R}^2) \times H^{s}_{\mathrm{loc}}(\mathbb{R}^2))$ for all $s < m$. ).

Since $m > 2$, $(\tilde{v}^{\varepsilon})$ is also precompact in $C([0,T]; C^1_{\mathrm{loc}}(\mathbb{R}^2) \times C^1_{\mathrm{loc}}(\mathbb{R}^2))$. Thus passing to the limit, we get that the limit function $\tilde{v} \in C([0,T]; V^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2))$ satisfies

$$\tilde{v}_t = \begin{pmatrix} -P(v \cdot \nabla v) + P(\theta f) \\ -(v \cdot \nabla \theta) \end{pmatrix}.$$

The first equation $P\left(v_t + (v \cdot \nabla v) - \theta f\right) = 0$ implies

$$v_t + (v \cdot \nabla v) - \theta f = \nabla p,$$

for some scalar function $p = p(x,t)$. Hence $\tilde{v}$ is a solution of the Boussinesq equations (2.1).

Proof of Uniqueness. Let

$$\tilde{v}_1 = \begin{pmatrix} v_1 \\ \theta_1 \end{pmatrix}, \quad \tilde{v}_2 = \begin{pmatrix} v_2 \\ \theta_2 \end{pmatrix}$$

be two solutions with the same initial data. If we set $v = v_1 - v_2$, $\theta = \theta_1 - \theta_2$, $p = p_1 - p_2$, then $v|_{t=0} = v_0 = 0$, $\theta|_{t=0} = \theta_0 = 0$. After subtracting corresponding terms, we get

$$\begin{cases} v_t + v_1 \cdot \nabla v + v \cdot \nabla v_2 = -\nabla p + \theta f, \\ \theta_t + v_1 \cdot \nabla \theta + v \cdot \nabla \theta_2 = 0. \end{cases}$$

Taking the $L^2$-inner product with $v$ and $\theta$ respectively, we obtain

$$\frac{d}{dt} \|\tilde{v}\|_0 \le \left( \|f\|_{L^\infty} + \|\nabla \tilde{v}_2\|_{L^\infty} \right) \|\tilde{v}\|_0.$$

Since $\tilde{v}_2 \in V^m\left(R^2\right) \times H^m\left(R^2\right)$ and $m > 2$,

$$\|\nabla v_2\|_{L^\infty} \le \|v_2\|_{H^m} \text{ and } \|\nabla \theta_2\|_{L^\infty} \le \|\theta_2\|_{H^m}.$$

Then Gronwall's inequality gives

$$\tilde{v} = \begin{pmatrix} v \\ \theta \end{pmatrix} \equiv 0$$

and the proof is complete. $\square$

# Blow-up Criterion

# Blow-up criterion

**Theorem 2 (Chae, Dongho; Nam, Hee-Seok 1997)**

Let the initial data $v_0 \in V^m\left(\mathbb{R}^2\right), \theta_0 \in H^m\left(\mathbb{R}^2\right)$ for some $m > 2$ and suppose that $f \in L^\infty\left([0, T]; W^{m,\infty}\left(\mathbb{R}^2\right)\right)$. Then we have:

$$\limsup_{t \nearrow T} \left(\|v(\cdot, t)\|_{H^m} + \|\theta(\cdot, t)\|_{H^m}\right) = \infty$$

if and only if

$$\int_0^T \|\nabla\theta(\cdot, \tau)\|_{L^\infty} d\tau = \infty.$$

Proof. (Necessity) Suppose $v$ and $\theta$ remain smooth on the time interval $[0, T]$, i.e.

$$\sup_{0\le t\le T} (\|v(\cdot,t)\|_{H^m} + \|\theta(\cdot,t)\|_{H^m}) \le C_T < \infty.$$

Since $m > 2$, by the Sobolev inequality,

$$\|\nabla\theta(\cdot,t)\|_{L^\infty} \le \|\theta(\cdot,t)\|_m \le C_T, \quad 0 \le t \le T. \tag{2.6}$$

This implies

$$\int_0^T \|\nabla\theta(\cdot,\tau)\|_{L^\infty} d\tau \le M_T < \infty.$$

(Sufficiency) Suppose that

$$\int_0^T \|\nabla\theta(\cdot,\tau)\|_{L^\infty} d\tau \le M_T.$$

We apply the curl operator to the first equation in (2.1).

$$\omega = \mathrm{curl}\, v = \frac{\partial}{\partial x_1} v_2 - \frac{\partial}{\partial x_2} v_1$$

can be read as a scalar function and we have

$$\omega_t + v \cdot \nabla \omega = (\nabla \theta \times f) \cdot e_3,$$

with the initial condition $\omega|_{t=0} = \omega_0 = \mathrm{curl}\, v_0$. Integrating over $[0, t]$, we obtain

$$\omega(\Psi_t(\alpha), t) = \omega_0(\alpha) + \int_0^t ((\nabla \theta \times f) \cdot e_3)(\Psi_s(\alpha), s)\, ds,$$

where $\Psi_t(\alpha)$ is the particle trajectories defined by the following ordinary differential equations:

$$\begin{cases} \dfrac{d}{dt}\Psi_t(\alpha) = v\left(\Psi_t(\alpha), t\right), \\ \Psi_t(\alpha)\big|_{t=0} = \alpha. \end{cases}$$

Using the generalised Minkowski inequality, we obtain

$$\|\omega(\cdot,t)\|_{L^p} \le \|\omega_0\|_{L^p} + \int_0^T \|\nabla\theta(\cdot,\tau)\|_{L^p}\|f(\cdot,\tau)\|_{L^\infty}d\tau. \tag{2.7}$$

Moreover, we can see that

$$\begin{aligned} \|\omega(\cdot,t)\|_{L^\infty} &\le \|\omega_0\|_{L^\infty} + \int_0^T \|\nabla\theta(\cdot,\tau)\|_{L^\infty}\|f(\cdot,\tau)\|_{L^\infty}d\tau \\ &\le C(\|v_0\|_{H^m}, \|\theta_0\|_{H^m}, \sup_{0\le t\le T}\|f(\cdot,t)\|_{L^\infty}, M_T). \end{aligned} \tag{2.8}$$

On the other hand, we apply the gradient operator to the second equation in (2.1) to get

$$\nabla\theta_t + (v\cdot\nabla)\nabla\theta = -\nabla v\nabla\theta.$$

Similarly to the above, we obtain

$$\begin{aligned}\|\nabla\theta(\cdot,t)\|_{L^p} &\le \|\nabla\theta_0\|_{L^p} + \int_0^T \|\nabla\theta(\cdot,\tau)\|_{L^\infty}\|\nabla v(\cdot,\tau)\|_{L^p}d\tau \\ &\le \|\nabla\theta_0\|_{L^p} + C_p\int_0^T \|\nabla\theta(\cdot,\tau)\|_{L^\infty}\|\omega(\cdot,\tau)\|_{L^p}d\tau,\end{aligned} \tag{2.9}$$

where we used the $\|\nabla v\|_{L^p} \le C_p\|\omega\|_{L^p}$ with $1 < p < \infty$ (Calderon-Zygmund inequality) in the last inequality. Combining (2.7) with (2.9), we obtain

$$\begin{aligned}&\|\omega(\cdot,t)\|_{L^p} + \|\nabla\theta(\cdot,t)\|_{L^p} \le \|\omega_0\|_{L^p} + \|\nabla\theta_0\|_{L^p} \\ &\quad + C_p\int_0^T \left(\|f(\cdot,\tau)\|_{L^\infty} + \|\nabla\theta(\cdot,\tau)\|_{L^\infty}\right)\left(\|\omega(\cdot,\tau)\|_{L^p} + \|\nabla\theta(\cdot,\tau)\|_{L^p}\right)d\tau.\end{aligned}$$

Then Gronwall's inequality gives

$$\begin{aligned}
&\|\omega(\cdot,t)\|_{L^p} + \|\nabla\theta(\cdot,t)\|_{L^p} \\
&\leq (\|\omega_0\|_{L^p} + \|\nabla\theta_0\|_{L^p}) \exp\left[C_p \int_0^t (\|f(\cdot,t)\|_{L^\infty} + \|\nabla\theta(\cdot,\tau)\|_{L^\infty})\, d\tau\right] \\
&\leq C\,(\|v_0\|_{H^m} + \|\theta_0\|_{H^m}) \exp\left[C_p \int_0^t (\|f(\cdot,\tau)\|_{L^\infty} + \|\nabla\theta(\cdot,\tau)\|_{L^\infty})\, d\tau\right] \\
&\leq C\left(\|v_0\|_{H^m}, \|\theta_0\|_{H^m}, \sup_{0\leq t\leq T}\|f(\cdot,t)\|_{L^\infty}, M_T, C_p\right),
\end{aligned} \tag{2.10}$$

where we used

$$\begin{aligned}
\|\omega_0\|_{L^p} &\leq C^{(p-2)/p}\,\|\omega_0\|_{L^2}^{(2/p)+(m-2)(p-2)/(m-1)p}\,\left\|D^{m-1}\omega_0\right\|_{L^2}^{(p-2)/(m-1)p} \\
&\leq C\,\|v_0\|_{H^m}, \quad C \geqq 1, \quad p \geqq 2,
\end{aligned}$$

and similarly

$$\|\nabla\theta_0\|_{L^p} \leq C\,\|\theta_0\|_{H^m}.$$

Now, recall the following well-known result (see e.g. Beale, J. T.; Kato, T.; Majda, A. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. Comm. Math. Phys.(1984)):

$$\|\nabla v(\cdot,t)\|_{L^\infty} \le C\left\{1+\left(1+\ln^+\|v(\cdot,t)\|_{H^m}\right)\|\omega(\cdot,t)\|_{L^\infty}+\|\omega(\cdot,t)\|_{L^p}\right\}.$$

Using (2.10) and (2.8) , the above inequality gives

$$\begin{aligned}&\|\nabla v(\cdot,t)\|_{L^\infty}\\ \le& C(\|v_0\|_{H^m}, \sup_{0\le t\le T}\|f(\cdot,t)\|_{L^\infty}, M_T, C_p)\left(1+\ln^+\|v(\cdot,t)\|_{H^m}\right)\end{aligned}$$

Applying (2.6) and above inequality to the $H^m$-energy estimate (2.5), we obtain

$$\begin{aligned}\frac{d}{dt}\|\tilde{v}(\cdot,t)\|_m \le C(\|v_0\|_{H^m}, \|\theta_0\|_{H^m}, \sup_{0\le t\le T}\|f(\cdot,t)\|_{W^{m,\infty}}, M_T, C_p)\\ \cdot\left(1+\ln^+\|\tilde{v}(\cdot,t)\|_m\right)\|\tilde{v}(\cdot,t)\|_m.\end{aligned}$$

Then Gronwall's inequality gives

$$\sup_{0\le t\le T} \|\tilde{v}^{\epsilon}\|_m \le C\left(\|v_0\|_{H^m}, \|\theta_0\|_{H^m}, \|f(\cdot,t)\|_{W^{m,\infty}}, M_T, C_p\right).$$

This concludes the proof. $\square$

# Application

As an application to the blow-up criterion, we prove global existence of smooth solutions in the case of zero external force. Suppose f = 0. Then the Boussinesq equations (2.1) become

$$\begin{cases} v_t + (v \cdot \nabla) v = -\nabla p \\ \theta_t + v \cdot \nabla \theta = 0 \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2 \\ \operatorname{div} v = 0 \\ v\big|_{t=0} = v_0, \quad \theta\big|_{t=0} = \theta_0, \quad x \in \mathbb{R}^2 \end{cases} \tag{2.11}$$

## Theorem 3

Let $v_0 \in V^m\left(\mathbb{R}^2\right)$, $\theta_0 \in H^m\left(\mathbb{R}^2\right)$, $m > 2$. Then the solution $(v, \theta)$ to the reduced Boussinesq equations (2.11) remains smooth globally in time.

Proof. First, we observe that

$$\|\omega(\cdot,t)\|_{L^\infty} \le \|\omega_0\|_{L^\infty}, \ \|\omega(\cdot t)\|_{L^2} \le \|\omega_0\|_{L^2}. \tag{2.12}$$

From the second equation in (2.11), we obtain

$$\begin{aligned}\frac{d}{dt}\|\theta(\cdot,t)\|_{H^m} &\le C\left(\|\nabla\theta(\cdot,t)\|_{L^\infty}\|v(\cdot,t)\|_{H^m} + \|\nabla v(\cdot,t)\|_{L^\infty}\|\theta(\cdot,t)\|_{H^m}\right)\\ &\le C\|v(\cdot,t)\|_{H^m}\|\theta(\cdot,t)\|_{H^m}.\end{aligned}$$

Then Gronwall's inequality gives us

$$\|\theta(\cdot,t)\|_{H^m} \le \|\theta_0\|_{H^m}\exp\left[C\int_0^t \|v(\cdot,\tau)\|_{H^m}d\tau\right]. \tag{2.13}$$

On the other hand, the first equation in (2.11) gives

$$\frac{d}{dt}\|v(\cdot,t)\|_{H^m} \le C\|\nabla v(\cdot,t)\|_{L^\infty}\|v(\cdot,t)\|_{H^m},$$

so that

$$\|v(\cdot,t)\|_{H^m} \le \|v_0\|_{H^m} \exp\left[C\int_0^t \|\nabla v(\cdot,\tau)\|_{L^\infty} d\tau\right], \qquad (2.14)$$

Applying (2.12) and (2.14) to

$$\|\nabla v(\cdot,t)\|_{L^\infty} \le C\{1+\left(1+\ln^+\|v(\cdot,t)\|_{H^m}\right)\|\omega(\cdot,t)\|_{L^\infty}+\|\omega(\cdot,t)\|_{L^p}\},$$

we obtain

$$\|\nabla v(\cdot,t)\|_{L^\infty} \le C\left(1+\int_0^t \|\nabla v(\cdot,\tau)\|_{L^\infty} d\tau\right).$$

Then Gronwall's inequality gives

$$\sup_{0\le t\le T} \|\nabla v(\cdot,t)\|_{L^\infty \le} \le \|\nabla v_0\|_{L^\infty} \exp[CT]. \qquad (2.15)$$

Combining (2.15), (2.14), (2.13) and using the fact $\|\nabla\theta\|_{L^\infty} \le \|\theta\|_{H^m}$, we obtain

$$\int_0^T \|\nabla\theta(\cdot,\tau)\|_{L^\infty} d\tau \le M_T < \infty, \quad \forall T > 0.$$

Then Theorem 3 gives the desired results and the proof is complete. $\square$

# Besov Space and its Properties

## Lemma (Bernstein inequality)

*Let $\mathcal{C}$ be a ring, $B$ a ball. A constant $C$ exists so that, for any non negative integer $k$, any smooth homogeneous function $\sigma$ of degree $m$, any couple of real $(a, b)$ so that $b \geq a \geq 1$ and any function $u$ of $L^a$, we have*

$$\text{Supp } \widehat{u} \subset \lambda B \Rightarrow \sup_{\alpha=k} \|\partial^\alpha u\|_{L^b} \leq C^{k+1} \lambda^{k+d\left(\frac{1}{a}-\frac{1}{b}\right)} \|u\|_{L^a};$$

$$\text{Supp } \widehat{u} \subset \lambda \mathcal{C} \Rightarrow C^{-k-1} \lambda^k \|u\|_{L^a} \leq \sup_{\alpha=k} \|\partial^\alpha u\|_{L^a} \leq C^{k+1} \lambda^k \|u\|_{L^a};$$

$$\text{Supp } \widehat{u} \subset \lambda \mathcal{C} \Rightarrow \|\sigma(D) u\|_{L^b} \leq C_{\sigma,m} \lambda^{m+d\left(\frac{1}{a}-\frac{1}{b}\right)} \|u\|_{L^a}$$

## Proposition (Dyadic Partition of Unity)

*Let us define by* $\mathcal{C}$ *the ring of center 0 , of small radius* $3/4$ *and great radius* $8/3$ *. It exists two radial functions* $\chi$ *and* $\varphi$ *the values of which are in the interval* $[0,1]$*, belonging respectively to* $\mathcal{D}(B(0,4/3))$ *and to* $\mathcal{D}(\mathcal{C})$ *such that*

$$\forall \xi \in \mathbb{R}^d, \ \chi(\xi) + \sum_{j\geq 0} \varphi\left(2^{-j}\xi\right) = 1,$$

$$\forall \xi \in \mathbb{R}^d \backslash \{0\}, \ \sum_{j\in\mathbb{Z}} \varphi\left(2^{-j}\xi\right) = 1,$$

$$|j-j'| \geq 2 \Rightarrow \operatorname{Supp}\varphi\left(2^{-j}\cdot\right) \cap \operatorname{Supp}\varphi\left(2^{-j'}\cdot\right) = \emptyset,$$
$$j \geq 1 \Rightarrow \operatorname{Supp}\chi \cap \operatorname{Supp}\varphi\left(2^{-j}\cdot\right) = \emptyset$$

## Proposition 3.1 (Dyadic Partition of Unity)

If $\widetilde{\mathcal{C}} = B(0, 2/3) + \mathcal{C}$, then $\widetilde{\mathcal{C}}$ is a ring and we have

$$\left|j - j'\right| \geq 5 \Rightarrow 2^{j'}\widetilde{\mathcal{C}} \cap 2^{j}\mathcal{C} = \emptyset,$$

$$\forall \xi \in \mathbb{R}^d,\ \frac{1}{3} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2\left(2^{-j}\xi\right) \leq 1,$$

$$\forall \xi \in \mathbb{R}^d \backslash \{0\},\ \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2\left(2^{-j}\xi\right) \leq 1.$$

# Notations

$$\Delta_{-1} u = \chi(D) u = \mathcal{F}^{-1}(\chi(\xi)\widehat{u}(\xi))$$

$$\text{if} \quad j \geq 0, \Delta_j u = \varphi\left(2^{-j} D\right) u,$$

$$\text{if} \quad j \leq -2, \Delta_j u = 0,$$

$$S_j u = \sum_{j' \leq j-1} \Delta_p u = \chi\left(2^{-j} D\right) u,$$

$$\text{if} \quad j \in \mathbb{Z},\ \dot{\Delta}_j u = \varphi\left(2^{-j} D\right) u, \quad \dot{S}_j u = \sum_{j' \leq j-1} \dot{\Delta}_p u.$$

## Definition

Let us denote by $\mathcal{S}'_h$ the space of tempered distribution such that

$$\lim_{j \to -\infty} \dot{S}_j u = 0 \text{ in } \mathcal{S}'.$$

## Proposition

*Tempered distribution $u$ belongs to $\mathcal{S}'_h$ if and only if, for any $\theta$ in $\mathcal{D}\left(\mathbb{R}^d\right)$ with value 1 near the origin, we have $\lim_{\lambda \to \infty} \theta(\lambda D)u = 0$ in $\mathcal{S}'$.*

## Definition (Homogeneous Besov Space)

Let $s$ be a real number and $(p, r)$ be in $[1, \infty]^2$. The homogeneous Besov space $\dot{B}^s_{p,r}$ consists of those distributions $u$ in $\mathcal{S}'_h$ such that

$$\|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \left( \sum_{j \in \mathbb{Z}} 2^{rjs} \left\| \dot{\Delta}_j u \right\|^r_{L^p} \right)^{\frac{1}{r}} < \infty.$$

## Proposition

*The space $\dot{B}^s_{p,r}$ endowed with $\| \cdot \|_{\dot{B}^s_{p,r}}$ is a normed space.*

### Theorem

*For any $p$ in $[2, \infty)$, $\dot{B}^0_{p,2}$ is continuously included in $L^p$ and $L^{p'}$ is continuously included in $\dot{B}^0_{p',2}$.*

### Theorem

*For any $p$ in $[1, 2]$, the space $\dot{B}^0_{p,p}$ is continuously included in $L^p$, and $L^{p'}$ is continuously included in $\dot{B}^0_{p',p'}$.*

### Theorem

*Let $1 \leq q < p < \infty$ and $\alpha$ be a positive real number. A constant $C$ exists such that*

$$\|f\|_{L^p} \leq C\|f\|^{1-\theta}_{\dot{B}^{-\alpha}_{\infty,\infty}} \|f\|^{\theta}_{\dot{B}^{\beta}_{q,q}} \quad \text{with} \quad \beta = \alpha\left(\frac{p}{q} - 1\right), \ \theta = \frac{q}{p}.$$

## Definition (Nonhomogeneous Besov space)

Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. The nonhomogeneous Besov space $B^s_{p,r}$ consists of all tempered distributions $u$ such that

$$\|u\|_{B^s_{p,r}} \overset{\text{def}}{=} \left\| \left( 2^{js} \left\| \Delta_j u \right\|_{L^p} \right)_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} < \infty.$$

## Proposition

*Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$. Then, for any real number $s$, the space $B^s_{p_1,r_1}$ is continuously embedded in $B^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$.*

## Theorem

*A constant $C$ exists which satisfies the following properties. If $s_1$ and $s_2$ are real numbers such that $s_1 < s_2$, $\theta \in (0,1)$, and $(p,r)$ is in $[1,\infty]$, then we have*

$$\|u\|_{B^{\theta s_1+(1-\theta s_2)}_{p,r}} \leq \|u\|^{\theta}_{B^{s_1}_{p,r}} \|u\|^{1-\theta}_{B^{s_2}_{p,r}},$$

*and*

$$\|u\|_{B^{\theta s_1+(1-\theta)s_2}_{p,1}} \leq \frac{C}{s_2 - s_1}\left(\frac{1}{\theta} + \frac{1}{1-\theta}\right) \|u\|^{\theta}_{B^{s_1}_{p,\infty}} \|u\|^{1-\theta}_{B^{s_2}_{p,\infty}}.$$

The Bony Decomposition:

Considering two tempered distributions u and v , we have

$$uv = \sum_{j',j} \Delta_{j'} u \Delta_j v.$$

## Definition (Paraproduct)

The nonhomogeneous paraproduct of v by u is defined by

$$T_u v \stackrel{\text{def}}{=} \sum_j S_{j-1} u \Delta_j v.$$

The nonhomogeneous remainder of $u$ and $v$ is defined by

$$R(u, v) = \sum_{|k-j|\leq 1} \Delta_k u \Delta_j v.$$

We have the following Bony decomposition:

$$uv = T_u v + T_v u + R(u, v).$$

We shall sometimes also use the following simplified decomposition:

$$uv = T_u v + T'_v u \quad \text{with} \quad T'_v u \stackrel{\text{def}}{=} \sum_j S_{j+2} v \Delta_j u.$$

# Full Dissipation Case

This section deals with the 2D viscous Boussinesq equations,

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + \theta \vec{e_2}, \\ \nabla \cdot u = 0, \\ \partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta, \end{cases} \tag{3.1}$$

where both $\nu$ and $\kappa$ are positive numbers. The global regularity can be established for this system of equations.

- J. R. Cannon, E. DiBenedetto, The initial value problem for the Boussinesq equations with data in Lp, Lecture Notes in Math., Vol. 771. Springer, Berlin, 1980, pp.129–144.

**Theorem 3.1**

Given an initial data $(u_0, \theta_0) \in H^2(\mathbb{R}^2)$. The 2D viscous Boussinesq equations (3.1) have a unique global classical solution $(u, \theta) \in C([0, \infty), H^2)$.

**Proof.** The proof of this result is almost trivial and similar to that for the 2D Navier-Stokes equations. It suffices to establish the global $H^1$ bound. First of all, we have the $L^2$-bounds

$$\|\theta\|_2^2 + 2\kappa \int_0^t \|\nabla\theta\|_2^2 d\tau = \|\theta_0\|_2^2,$$

$$\|u\|_2^2 + 2\nu \int_0^t \|\nabla u\|_2^2 d\tau \le (\|u_0\|_2 + t\|\theta_0\|_2)^2.$$

It follows from the vorticity equation

$$\partial_t\omega + u\cdot\nabla\omega = \nu\Delta\omega + \partial_{x_1}\theta,$$

that

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2 + \nu\|\nabla\omega\|_2^2 \le \|\theta\|_2^2\|\partial_{x_1}\omega\|_2^2 \le \frac{\nu}{2}\|\nabla\omega\|_2^2 + \frac{1}{2\nu}\|\theta_0\|_2^2,$$

$$\|\omega\|_2^2 + \nu \int_0^t \|\nabla\omega\|_2^2 d\tau \le \|\omega_0\|_2^2 + \frac{1}{\nu} t \|\theta_0\|_2^2.$$

By the equation for $\theta$,

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|_2^2 + \kappa\|\Delta\theta\|_2^2 \le \int |\nabla u||\nabla\theta|^2 dx \le C\,\|\nabla u\|_3\|\nabla\theta\|_3^2. \quad (3.2)$$

Applying the Gagliardo-Nirenberg inequality

$$\|f\|_{L^3(\mathbb{R}^d)} \le C\,\|f\|_{L^2(\mathbb{R}^d)}^{1-\frac{d}{6}}\,\|\nabla f\|_{L^2(\mathbb{R}^d)}^{\frac{d}{6}}, \quad (3.3)$$

with $d = 2$, namely $\|f\|_3 \le C\,\|f\|_2^{2/3}\,\|\nabla f\|_2^{1/3}$, we obtain

$$\|\nabla u\|_3^3 \le C\,\|\nabla u\|_2^2\,\|\nabla^2 u\|_2 \le \frac{\nu}{2}\|\nabla^2 u\|_2^2 + \frac{C}{\nu}\|\nabla u\|_2^2\,\|\nabla u\|_2^2.$$

Therefore,

$$C\,\|\nabla u\|_3\|\nabla\theta\|_3^2 \le \frac{\nu}{2}\|\nabla^2 u\|_2^2 + \frac{\kappa}{2}\|\nabla^2\theta\|_2^2 + \frac{C}{\nu}\|\nabla u\|_2^2\,\|\nabla u\|_2^2 + \frac{C}{\kappa}\|\nabla\theta\|_2^2\,\|\nabla\theta\|_2^2.$$

Inserting this inequality in (3.2) and applying the integrability

$$\int_0^\infty \|\nabla u\|_2^2\,d\tau < \infty, \quad \int_0^\infty \|\nabla\theta\|_2^2\,d\tau < \infty,$$

we have, for any $T > 0$,

$$\|\nabla\theta(T)\|_2^2 + \kappa\int_0^T \|\Delta\theta\|_2^2 d\tau \le C(T).$$

$H^2$ norm can be obtained through a similar procedure.

We would like to point out that (3.3) depends on the dimension $d$ and a similar procedure does not yield a global $H^1$ bound in the 3D case. When $d = 3$, we have

$$\|\nabla u\|_{L^3(\mathbb{R}^3)}^3 \leq C\, \|\nabla u\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}} \|\nabla^2 u\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}} \leq \frac{\nu}{2}\|\nabla^2 u\|_{L^2(\mathbb{R}^3)}^2 + \frac{C}{\nu}\|\nabla u\|_{L^2(\mathbb{R}^3)}^4$$

But now $\|\nabla u\|_{L^2(\mathbb{R}^3)}^4$ is no longer time integrable. This completes the proof of Theorem 3.1. $\square$

# Partial Dissipation Case

Consider the Cauchy problem of zero diffusivity Boussinesq equations:

$$(B_1)\begin{cases}\dfrac{\partial v}{\partial t}+(v\cdot\nabla)v=-\nabla p+\nu\Delta v+\theta e_2,\\ \dfrac{\partial\theta}{\partial t}+(v\cdot\nabla)\theta=0,\quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^2,\\ \operatorname{div} v=0,\\ v(0,x)=v_0(x),\quad \theta(0,x)=\theta_0(x).\end{cases}$$

The global regularity was obtained by T. Hou and C. Li, Global well-posedness of the viscous Boussinesq equations, Discrete and Cont. Dyn. Syst. 12 (2005) and by D. Chae, Global regularity for the 2D Boussinesq equations with partial viscosity terms, Advances in Math. 203 (2006) .

The result can be stated as follows.

**Theorem 3.2 (T. Hou and C. Li 2005, D. Chae 2006)**

Let $\nu > 0$ be fixed, and $\operatorname{div} v_0 = 0$. Let $m > 2$ be an integer, and $(v_0, \theta_0) \in H^m(\mathbb{R}^2)$. Then, there exists a unique solution $(v, \theta)$ with $\theta \in C([0,\infty); H^m(\mathbb{R}^2))$and $v \in C([0,\infty); H^m(\mathbb{R}^2)) \cap L^2(0, T; H^{m+1}(\mathbb{R}^2))$ of the system $(B_1)$.

We also write down the zero viscosity Boussinesq equations

$$(B_2)\begin{cases}\dfrac{\partial v}{\partial t}+(v\cdot\nabla)v=-\nabla p+\theta e_2,\\ \dfrac{\partial \theta}{\partial t}+(v\cdot\nabla)\theta=\kappa\Delta\theta, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^2,\\ \operatorname{div} v=0,\\ v(0,x)=v_0(x), \quad \theta(0,x)=\theta_0(x).\end{cases}$$

The following is the global regularity result on $(B_2)$.

**Theorem 3.3 (D. Chae 2006)**

Let $\kappa>0$ be fixed, and $\operatorname{div} v_0=0$. Let $m>2$ be an integer. Let $m>2$ be an integer, and $(v_0,\theta_0)\in H^m\left(\mathbb{R}^2\right)$. Then, there exists unique solutions $(v,\theta)$ with $v\in C\left([0,\infty);H^m\left(\mathbb{R}^2\right)\right)$ and $\theta\in C\left([0,\infty);H^m\left(\mathbb{R}^2\right)\right)\cap L^2\left(0,T;H^{m+1}\left(\mathbb{R}^2\right)\right)$ of the system $(B_2)$.

# Proof of Theorem 3.2 ($\nu > 0, \kappa = 0$)

Let $T > 0$ be a given fixed time. From the second equation of $(B_1)$ we immediately have

$$\|\theta(t)\|_{L^p} \leqslant \|\theta_0\|_{L^p}, \quad \forall t \in [0, T], \quad p \in [1, \infty].$$

Taking $L^2$ inner product the first equation of $(B_1)$ with $v$, we have,

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 + \nu\|\nabla v\|_{L^2}^2 \leqslant \|\theta\|_{L^2}\|v\|_{L^2}.$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 \leqslant \|\theta\|_{L^2}\|v\|_{L^2} \leqslant \|\theta_0\|_{L^2}\|v\|_{L^2},$$

Hence, $\frac{d}{dt}\|v\|_{L^2} \leqslant \|\theta_0\|$, and we obtain

$$\|v(t)\|_{L^2} \leqslant \|v_0\|_{L^2} + \|\theta_0\|_{L^2} T, \quad \forall t \in [0, T]. \tag{3.4}$$

Taking the operation curl on both sides of the first equation of $(B_1)$, we obtain

$$\omega_t + (v \cdot \nabla)\omega = -\theta_{x_1} ? + \nu \Delta \omega,$$

where $\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1$. Let $p \geqslant 2$. Multiplying above equation by $\omega|\omega|^{p-2}$ and integrating it over $\mathbb{R}^2$, we find,

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|\omega|^p dx + (p-1)\nu\int_{\mathbb{R}^2}|\nabla\omega|^2|\omega|^{p-2}dx \\
&\quad = \frac{1}{p}\int_{\mathbb{R}^2}(v\cdot\nabla)|\omega|^p dx - \int_{\mathbb{R}^2}\theta_{x_1}\omega|\omega|^{p-2}dx \\
&\quad = -\frac{1}{p}\int_{\mathbb{R}^2}\operatorname{div} v|\omega|^p dx + (p-1)\int_{\mathbb{R}^2}\theta\omega x_{x_1}|\omega|^{p-2}dx \\
&\quad \leqslant \frac{(p-1)\nu}{2}\int_{\mathbb{R}^2}|\nabla\omega|^2|\omega|^{p-2}dx + \frac{(p-1)}{2\nu}\int_{\mathbb{R}^2}\theta^2|\omega|^{p-2}dx \\
&\quad \leqslant \frac{(p-1)\nu}{2}\int_{\mathbb{R}^2}|\nabla\omega|^2|\omega|^{p-2}dx + \frac{(p-1)}{2\nu}\|\theta\|_{L^p}^2\|\omega\|_{L^p}^{p-2}.
\end{aligned}
$$

Carrying over the term $\frac{(p-1)\nu}{2}\int_{\mathbb{R}^2}|\nabla\omega|^2|\omega|^{p-2}dx$ to the left-hand side, we find

$$
\frac{1}{p}\frac{d}{dt}\|\omega\|_{L^p}^p + \frac{(p-1)\nu}{2}\int_{\mathbb{R}^2}|\nabla\omega|^2|\omega|^{p-2}dx \leqslant \frac{(p-1)}{2\nu}\|\theta\|_{L^p}^2\|\omega\|_{L^p}^{p-2}. \tag{3.5}
$$

For $p = 2$, in particular, after integration over $[0, T]$ we obtain

$$\|\omega(t)\|_{L^2}^2 + \nu \int_0^T \|\nabla\omega(s)\|_{L^2}^2 ds \leqslant 2\|\omega_0\|_{L^2}^2 + \frac{2}{\nu}\|\theta_0\|_{L^2}^2 T, \quad \forall t \in [0, T].$$

Hence, we find that, by Hölder's inequality,

$$\begin{aligned}\int_0^T \|\nabla\omega(s)\|_{L^2} ds &\leqslant C\sqrt{T}\left(\int_0^T \|\nabla\omega(s)\|_{L^2}^2 ds\right)^{1/2} \\ &\leqslant C\|\omega_0\|_{L^2}\sqrt{T} + C\|\theta_0\|_{L^2} T, \quad \forall t \in [0, T].\end{aligned} \tag{3.6}$$

On the other hand, from (3.5) , we have for $p \in [2, \infty)$

$$\|\omega(t)\|_{L^p}^2 \leqslant \|\omega_0\|_{L^p}^2 + \frac{(p-1)}{\nu}\|\theta_0\|_{L^p}^2 T \leqslant \left(\|\omega_0\|_{L^p} + \frac{\sqrt{p-1}}{\sqrt{\nu}}\|\theta_0\|_{L^p}\sqrt{T}\right) \tag{3.7}$$

and

$$\|\omega(t)\|_{L^p} \leqslant \|\omega_0\|_{L^p} + \frac{\sqrt{p-1}}{\sqrt{\nu}} \|\theta_0\|_{L^p} \sqrt{T}, \quad \forall t \in [0, T], \quad p \in [2, \infty). \tag{3.8}$$

Recall the Gagliardo-Nirenberg interpolation inequality in $\mathbb{R}^2$.

$$\|f\|_{L^\infty} \leqslant C \|f\|_{L^p}^{\frac{p-2}{2p-2}} \|Df\|_{L^p}^{\frac{p}{2p-2}}, \quad f \in W^{1,p}\left(\mathbb{R}^2\right), \quad p > 2. \tag{3.9}$$

By this and the Calderon-Zygmund inequality combined with estimates (3.4) and (3.8) for $p \in (2, \infty)$ we find

$$\begin{aligned} \|v(t)\|_{L^\infty} &\leqslant C \|v(t)\|_{L^2}^{\frac{p-2}{2p-2}} \|\nabla v(t)\|_{L^p}^{\frac{p}{2p-2}} \leqslant C \|v(t)\|_{L^2}^{\frac{p-2}{2p-2}} \|\omega(t)\|_{L^p}^{\frac{p}{2p-2}} \\ &\leqslant C\left(v_0, \theta_0, T, \nu, p\right), \quad \forall t \in [0, T]. \end{aligned} \tag{3.10}$$

# $W^{2,p}$ estimate for v

We take the derivative operation $D = (\partial_{x_1}, \partial_{x_2})$ on the equation of $\omega$, and then take $L^2$ inner product with $D\omega|D\omega|^{p-2}$, $p > 2$.

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|D\omega\|_{L^p}^p + (p-1)\nu\int_{\mathbb{R}^2}\left|D^2\omega\right|^2|D\omega|^{p-2}dx \\
&= -\int_{\mathbb{R}^2}[D(v\cdot\nabla)\omega]D\omega|D\omega|^{p-2}dx - \int_{\mathbb{R}^2}D\theta_{x_1}D\omega|D\omega|^{p-2}dx \\
&=(p-1)\int_{\mathbb{R}^2}[(v\cdot\nabla)\omega]D^2\omega|D\omega|^{p-2}dx + (p-1)\int_{\mathbb{R}^2}\theta_{x_1}D^2\omega|D\omega|^{p-2}dx \\
&\leq\frac{(p-1)\nu}{4}\int_{\mathbb{R}^2}\left|D^2\omega\right|^2|D\omega|^{p-2}dx + \frac{(p-1)}{\nu}\int_{\mathbb{R}^2}|v(x)|^2|D\omega|^p dx \\
&\quad + \frac{(p-1)\nu}{4}\int_{\mathbb{R}^2}\left|D^2\omega\right|^2|D\omega|^{p-2}dx + \frac{(p-1)}{\nu}\int_{\mathbb{R}^2}|\nabla\theta|^2|D\omega|^{p-2}dx,
\end{aligned}$$

We have

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|D\omega\|_{L^p}^p + \frac{(p-1)\nu}{2}\int_{\mathbb{R}^2}\left|D^2\omega\right|^2|D\omega|^{p-2}dx \\
&\quad\leqslant \frac{(p-1)}{\nu}\int_{\mathbb{R}^2}|v(x)|^2|D\omega|^p dx + \frac{(p-1)}{\nu}\int_{\mathbb{R}^2}|\nabla\theta|^2|D\omega|^{p-2}dx \\
&\quad\leqslant \frac{(p-1)}{\nu}\|v\|_{L^\infty}^2\|D\omega\|_{L^p}^p + \frac{2(p-1)}{p\nu}\|\nabla\theta\|_{L^p}^p + \frac{(p-1)(p-2)}{p\nu}\|D\omega\|_{L^p}^p,
\end{aligned}$$

where we used Young's inequality, $a^2b^{p-2} \leqslant \frac{2}{p}a^p + \frac{p-2}{p}b^p$ for $p \geqslant 2$. Recalling the estimate of $\|v(t)\|_{L^\infty}$ in (3.10), we find that

$$\frac{d}{dt}\|D\omega\|_{L^p}^p \leqslant C\|D\omega\|_{L^p}^p + C\|\nabla\theta\|_{L^p}^p \quad \forall t\in[0,T], \tag{3.11}$$

where $C = C\left(v_0, \theta_0, T, \nu, p\right)$.

Now taking $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$ to the second equation of $(B_1)$, we obtain

$$\nabla^{\perp}\theta_t + (v\cdot\nabla)\nabla^{\perp}\theta = \nabla^{\perp}\theta\cdot\nabla v.$$

Taking $L^2$ inner product (2.12) with $\nabla^{\perp}\theta\left|\nabla^{\perp}\theta\right|^{p-2}$, we deduce, after integration by part, that

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla\theta|^p dx &= -\frac{1}{p}\int_{\mathbb{R}^2}(v\cdot\nabla)|\nabla\theta|^p dx + \int_{\mathbb{R}^2}\left(\nabla^{\perp}\theta\cdot\nabla\right)v\cdot\nabla^{\perp}\theta|\nabla\theta|^{p-2}dx\\ &\leqslant \int_{\mathbb{R}^2}|\nabla v||\nabla\theta|^p dx.\end{aligned}$$

Hence, for $p > 2$ we have

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla\theta\|_{L^p}^p \\
&\quad\leqslant p\|\nabla v\|_{L^\infty}\|\nabla\theta\|_{L^p}^p \\
&\quad\leqslant C\left(1+\|\nabla v\|_{L^2}+\left\|D^2 v\right\|_{L^2}\right)\left[1+\log^+\left(\left\|D^2 v\right\|_{L^p}\right)\right]\|\nabla\theta\|_{L^p}^p \\
&\quad\leqslant C\left(1+\|\omega\|_{L^2}+\|D\omega\|_{L^2}\right)\left[1+\log^+\left(\|D\omega\|_{L^p}^p+\|\nabla\theta\|_{L^p}^p\right)\right]\|\nabla\theta\|_{L^p}^p, \\
&\quad\leqslant C\left(1+\|D\omega\|_{L^2}\right)\left[1+\log^+\left(\|D\omega\|_{L^p}^p+\|\nabla\theta\|_{L^p}^p\right)\right]\|\nabla\theta\|_{L^p}^p.
\end{aligned}
\tag{3.12}
$$

where $C = (v_0, \theta_0, T, \nu, p)$, and we used the following form of the Brezis-Wainger inequality.

(H. Brezis, S. Wainger, A note on limiting cases of Sobolev embeddings and convolution inequalities, Comm. Partial Differential Equations 5 (7) (1980) )

$$\|f\|_{L^\infty} \leqslant (1+\|\nabla f\|_{L^2}) \left[1+\log^+ (\|\nabla f\|_{L^p})\right]^{\frac{1}{2}} + C\|f\|_{L^2}. \quad (3.13)$$

for $f \in L^2(\mathbb{R}^2) \cap W^{1,p}(\mathbb{R}^2)$, which holds for $p > 2$, Adding (3.11) and (3.12) together, and setting $X(t) = \|\nabla\theta\|_{L^p}^p + \|D\omega\|_{L^p}^p$, we find that

$$\frac{dX}{dt} \leqslant C(1+\|D\omega(t)\|_{L^2})\left(1+\log^+ X\right) X.$$

for all $t \in [0, T]$, where $C = C(v_0, \theta_0, T, \nu, p)$.

By Gronwall's lemma we have

$$X(t) \leqslant X(0) \exp\left[\exp\left\{CT + C\int_0^T \|D\omega(s)\|_{L^2} ds\right\}\right], \quad \forall t \in [0, T],$$

which, combined with estimate (3.6), implies that for $p > 2$

$$\|D\omega(t)\|_{L^p} \leqslant C\left(v_0, \theta_0, T, \nu, p\right), \quad \forall t \in [0, T]. \tag{3.14}$$

By the Gagliardo-Nirenberg interpolation inequality (3.9) we have

$$\begin{aligned} \|\nabla v(t)\|_{L^\infty} &\leqslant C\|\nabla v(t)\|_{L^2}^{\frac{p-2}{2p-2}} \left\|D^2 v(t)\right\|_{L^p}^{\frac{p}{2p-2}} \leqslant C\|\omega(t)\|_{L^2}^{\frac{p-2}{2p-2}} \|D\omega(t)\|_{L^p}^{\frac{p}{2p-2}} \\ &\leqslant C\left(v_0, \theta_0, T, \nu, p\right) \quad \forall t \in [0, T], \quad p \in (2, \infty], \end{aligned} \tag{3.15}$$

where we used the estimates (3.7) and (3.14).

Recall that the $L^p$ estimate of $\nabla\theta$ (3.12), we have

$$\frac{d}{dt}\|\nabla\theta\|_{L^p} \leqslant \|\nabla v\|_{L^\infty}\|\nabla\theta\|_{L^p},$$

and by Gronwall's lemma

$$\|\nabla\theta(t)\|_{L^p} \leqslant \|\nabla\theta_0\|_{L^p} \exp\left(\int_0^t \|\nabla v(s)\|_{L^\infty} ds\right). \tag{3.16}$$

Then, applying the following $L^p$ interpolation inequality to (3.16):

$$\|f\|_{L^p} \leqslant \|f\|_{L^2}^{\frac{2}{p}}\|f\|_{L^\infty}^{1-\frac{2}{p}}, \quad 2 \leqslant p \leqslant \infty,$$

we obtain

$$\|\nabla\theta(t)\|_{L^p} \leqslant \|\nabla\theta_0\|_{L^2}^{\frac{2}{p}}\|\nabla\theta_0\|_{L^\infty}^{1-\frac{2}{p}} \exp\left(\int_0^T \|\nabla v(s)\|_{L^\infty}\right),$$

we obtain

$$\|\nabla\theta(t)\|_{L^p} \leqslant (1+\|\nabla\theta_0\|_{L^2})(1+\|\nabla\theta_0\|_{L^\infty}) \exp\left(\int_0^T \|\nabla v(s)\|_{L^\infty}\right),$$

Passing first $p \to \infty$, we have

$$\begin{aligned}\|\nabla\theta(t)\|_{L^\infty} &\leqslant (1+\|\nabla\theta_0\|^2_{L^2\cup L^\infty}) \exp\left(\int_0^T \|\nabla v(s)\|_{L^\infty} ds\right) \\ &\leqslant C \quad \forall t \in [0, T],\end{aligned}$$

where $C = C(\|v_0\|_{H^m}, \|\theta_0\|_{H^m}, T, \nu)$, and we used the estimate of $\|\nabla v\|_{L^\infty}$(3.15).

Since we have the embedding, $H^m\left(\mathbb{R}^2\right) \hookrightarrow W^{2,p}\left(\mathbb{R}^2\right)$, for all $m > 2$ and $p > 2$ we attained estimate

$$\int_0^T \|\nabla\theta(\cdot,\tau)\|_{L^\infty} d\tau < \infty.$$

for any given $T \in (0, \infty)$ and for all $v_0, \theta_0 \in H^m\left(\mathbb{R}^2\right)$ with $m > 2$, the proof is completed. $\square$

# Proof of Theorem 3.3 ($\nu = 0, \kappa > 0$)

Similar to the previous subsection, in order to prove the global regularity of $(B_2)$, we have only to prove estimate $L^2_t L^\infty_x$ of $\nabla\theta$ for the classical solution of $(B_2)$ for all $T \in (0, \infty)$.
First, we can easily get the $L^2$ estimates for $\theta, v$ and $\omega$,

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2} + \kappa\|\nabla\theta\|^2_{L^2} = 0.$$

Integrating this over $[0, T]$ we have

$$\frac{1}{2}\|\theta(t)\|^2_{L^2} + \int_0^T \|\nabla\theta\|^2_{L^2} dt \leqslant \frac{1}{2}\|\theta_0\|^2_{L^2} \quad \forall t \in [0, T]. \qquad (3.17)$$

For $v$,

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 = \int_{\mathbb{R}^2} \theta e_2 \cdot v dx \leqslant \|\theta\|_{L^2}\|v\|_{L^2}.$$

Combining this with (3.17), we easily obtain

$$\|v(t)\|_{L^2} \leqslant \|v_0\|_{L^2} + \int_0^T \|\theta(s)\|_{L^2} ds = \|v_0\|_{L^2} + T\,\|\theta_0\|_{L^2},$$

for all $t \in [0, T]$. Taking the curl of the first equation of $(B_2)$, we have

$$\omega_t + (v \cdot \nabla)\omega = -\theta_{x_1}. \tag{3.18}$$

Taking $L^2$ inner product with $\omega$, and integrating by part, we deduce

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 \leqslant \int_{\mathbb{R}^2} |\nabla\theta||\omega| dx \leqslant \|\nabla\theta\|_{L^2}\|\omega\|_{L^2},$$

and

$$\frac{d}{dt}\|\omega\|_{L^2} \leqslant \|\nabla\theta\|_{L^2}.$$

Hence, using estimate (3.17), we derive

$$\begin{aligned}
\|\omega(t)\|_{L^2} &\leqslant \int_0^T \|\nabla\theta\|_{L^2} dt + \|\omega_0\|_{L^2} \\
&\leqslant T^{\frac{1}{2}} \left( \int_0^T \|\nabla\theta\|_{L^2}^2 dt \right)^{1/2} + \|\omega_0\|_{L^2} \\
&\leqslant \frac{T^{\frac{1}{2}}}{\sqrt{2}} \|\theta_0\|_{L^2} + \|\omega_0\|_{L^2}, \quad \forall t \in [0, T].
\end{aligned}$$

$W^{1,p}$ estimate for $(\theta, v)$

Using operation $\nabla^{\perp}$ on the second equation of $(B_2)$, we have

$$\nabla^{\perp}\theta + (v \cdot \nabla)\nabla^{\perp}\theta = \left(\nabla^{\perp}\theta \cdot \nabla\right) v + \kappa\Delta\nabla^{\perp}\theta.$$

We now take scalar product in $L^2$ by $\nabla^{\perp}\theta \left|\nabla^{\perp}\theta\right|^{p-2}$, $p > 2$;

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|\nabla^{\perp}\theta\|_{L^p}^{p} + (p-1)\kappa \int_{\mathbb{R}^2} |D^2\theta|^2|\nabla^{\perp}\theta|^{p-2}dx \\
&\quad= \int_{\mathbb{R}^2}(\nabla^{\perp}\theta\nabla)v\nabla^{\perp}\theta|\nabla^{\perp}\theta|^{p-2}dx \\
&\quad= -(p-1)\int_{\mathbb{R}^2} v\cdot(\nabla^{\perp}\theta\cdot\nabla)\nabla^{\perp}\theta|\nabla^{\perp}\theta|^{p-2}dx \\
&\quad\leqslant (p-1)\int_{\mathbb{R}^2}|v||\nabla^{\perp}\theta||D^2\theta||\nabla^{\perp}\theta|^{p-2}dx \\
&\quad\leqslant \frac{(p-1)}{2\kappa}\int_{\mathbb{R}^2}|v|^2|\nabla^{\perp}\theta|^p dx + \frac{(p-1)\kappa}{2}\int_{\mathbb{R}^2}|D^2\theta|^2|\nabla^{\perp}\theta|^{p-2}dx.
\end{aligned}$$

We carry over the second term to the left-hand side to have

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla\theta\|_{L^p}^p + \frac{p(p-1)\kappa}{2}\int_{\mathbb{R}^2}\left|D^2\theta\right|^2|\nabla\theta|^{p-2}dx \\
&\quad\leqslant \frac{(p-1)p}{2\kappa}\|v\|_{L^\infty}^2\|\nabla\theta\|_{L^p}^p \\
&\quad\leqslant C\left(1+\|v\|_{L^2}+\|\nabla v\|_{L^2}\right)^2\left(1+\log^+\left(\|\nabla v\|_{L^p}^p\right)\right)\|\nabla\theta\|_{L^p}^p \\
&\quad\leqslant C\left(1+\|v\|_{L^2}+\|\omega\|_{L^2}^2\right)\left[1+\log^+\left(\|\omega\|_{L^p}^p+\|\nabla\theta\|_{L^p}^p\right)\right]\|\nabla\theta\|_{L^p}^p \\
&\quad\leqslant C\left[1+\log^+\left(\|\omega\|_{L^p}^p+\|\nabla\theta\|_{L^p}^p\right)\right]\|\nabla\theta\|_{L^p}^p,
\end{aligned} \tag{3.19}
$$

where we applied the Brezis-Wainger inequality (3.13).

On the other hand, taking $L^2$ inner product (3.18) with $\omega|\omega|^{p-2}$, we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|\omega\|_{L^p}^p + \frac{1}{p}\int_{\mathbb{R}^2}(v\cdot\nabla)|\omega|^p dx &= -\int_{\mathbb{R}^2}\theta_{x_2}\omega|\omega|^{p-2}dx \\ &\leqslant \int_{\mathbb{R}^2}|\nabla\theta||\omega|^{p-1}dx \\ &\leqslant \frac{1}{p}\|\nabla\theta\|_{L^p}^p + \frac{(p-1)}{p}\|\omega\|_{L^p}^p, \end{aligned} \tag{3.20}$$

Adding (3.20) to (3.19) , and setting $X(t) = \|\nabla\theta(t)\|_{L^p}^p + \|\omega\|_{L^p}^p$, we have

$$\frac{d}{dt}X(t) \leqslant C(1+\log X(t))X(t), \quad \forall t\in[0,T].$$

The Gronwall lemma provides us with

$$X(t)\leqslant X(0)e^{e^{CT}}, \quad \forall t\in[0,T].$$

Hence,

$$\|\nabla\theta(t)\|_{L^p}^p + \|\omega\|_{L^p}^p \leqslant C\left(v_0, \theta_0, T, p, \kappa\right).$$

We also note that similar to (3.10), combined with above inequalities implies that

$$\begin{aligned}
\|v(t)\|_{L^\infty} &\leqslant C\|v(t)\|_{L^2}^{\frac{p-2}{2p-2}}\|\nabla v(t)\|_{L^p}^{\frac{p}{2p-2}} \\
&\leqslant C\|v(t)\|_{L^2}^{\frac{p-2}{2p-2}}\|\omega(t)\|_{L^p}^{\frac{p}{2p-2}} \\
&\leqslant C\left(v_0, \theta_0, T, \kappa, p\right), \quad \forall t \in [0, T].
\end{aligned}$$

$W^{2,p}$ estimate for $\theta$

Taking operation $D^2$ on the second equation of $(B_2)$, and then taking $L^2$ inner product of this with $D^2\theta\left|D^2\theta\right|^{p-2}$, $p>2$, we have after integration by part

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\left\|D^2\theta\right\|_{L^p}^p+(p-1)\kappa\int_{\mathbb{R}^2}\left|D^3\theta\right|^2\left|D^2\theta\right|^{p-2}dx\\
&=-\int_{\mathbb{R}^2}D^2(v\cdot\nabla)\theta D^2\theta\left|D^2\theta\right|^{p-2}=(p-1)\int_{\mathbb{R}^2}D[(v\cdot\nabla)\theta]D^3\theta\left|D^2\theta\right|^{p-2}dx\\
&=(p-1)\int_{\mathbb{R}^2}Dv\cdot D\theta D^3\theta\left|D^2\theta\right|^{p-2}dx+(p-1)\int_{\mathbb{R}^2}[(v\cdot\nabla)D\theta]D^3\theta\left|D^2\theta\right|^{p-2}dx\\
&\leqslant\frac{(p-1)}{\kappa}\|\nabla\theta\|_{L^\infty}^2\int_{\mathbb{R}^2}|\nabla v|^2\left|D^2\theta\right|^{p-2}dx+\frac{(p-1)\kappa}{4}\int_{\mathbb{R}^2}\left|D^3\theta\right|^2\left|D^2\theta\right|^{p-2}dx\\
&\quad+\frac{(p-1)}{\kappa}\|v\|_{L^\infty}^2\int_{\mathbb{R}^2}\left|D^2\theta\right|^p dx+\frac{(p-1)\kappa}{4}\int_{\mathbb{R}^2}\left|D^3\theta\right|^2\left|D^2\theta\right|^{p-2}dx.
\end{aligned}
$$

Carrying over the terms, $\frac{(p-1)\kappa}{4}\int_{\mathbb{R}^2}\left|D^3\theta\right|^2\left|D^2\theta\right|^{p-2}dx$ to the left-hand side, we derive

$$\begin{aligned}\frac{d}{dt}\left\|D^2\theta\right\|_{L^p}^p &\leqslant C\|\nabla\theta\|_{L^\infty}^2\|\nabla v\|_{L^p}^2\left\|D^2\theta\right\|_{L^p}^{p-2}+C\|v\|_{L^\infty}^2\left\|D^2\theta\right\|_{L^p}^p\\ &\leqslant C\|\nabla\theta\|_{L^p}^{\frac{2p-4}{2p-2}}\|\omega\|_{L^p}^2\left\|D^2\theta\right\|_{L^p}^{p-\frac{2p-4}{2p-2}}+C\|v\|_{L^\infty}^2\left\|D^2\theta\right\|_{L^p}^p\\ &\leqslant C+C\left\|D^2\theta\right\|_{L^p}^p,\end{aligned}$$

where we used the Gagliardo-Nirenberg interpolation inequality (3.9)(note that $p-\frac{2p-4}{2p-2}<p$ when $p>2$).

Thanks to Gronwall's lemma, we have the estimate

$$\left\|D^2\theta(t)\right\|_{L^p} \leqslant C\left(v_0, \theta_0, T, p, \kappa\right), \quad \forall t \in [0, T], \quad \forall p > 2.$$

Using the interpolation inequality (3.15) as previously, we obtain that

$$\|\nabla\theta(t)\|_{L^\infty} \leqslant C, \quad \forall t \in [0, T],$$

where $C = C\left(\|v_0\|_{W^{2,p}}, \|\theta_0\|_{W^{2,p}}, p, \kappa\right)$. Similar to the proof of Theorem 3.2, we have the embedding, $H^m\left(\mathbb{R}^2\right) \hookrightarrow W^{2,p}\left(\mathbb{R}^2\right)$, for all $m > 2$ and $p > 2$, and thus we attained estimate

$$\int_0^T \|\nabla\theta(\cdot, \tau)\|_{L^\infty} d\tau < \infty,$$

which complete the proof. $\square$

# 2D Boussinesq Equations with Fractional Dissipation

We consider the Cauchy problem of 2D fractional diffusion Boussinesq equations for an incompressible fluid flows in $\mathbb{R}^2$

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u + \nu (-\Delta)^{\alpha} u + \nabla P = \theta e_2, \\ \dfrac{\partial \theta}{\partial t} + (u \cdot \nabla)\theta + \kappa (-\Delta)^{\beta} \theta = 0, \\ \operatorname{div} u = 0, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{3.21}$$

where $\alpha$, $\beta \in (0,1)$, and $(-\Delta)^{\alpha}$ is the pseudodifferential operator defined via the Fourier transform

$$\widehat{(-\Delta)^{\alpha} v}(\xi) = |\xi|^{2\alpha} \widehat{v}(\xi).$$

In the following, for simplicity, we denote

$$\Lambda = (-\Delta)^{1/2}.$$

## Theorem 3.4 (Xiaojing Xu 2010)

Let $\nu > 0$, $\kappa > 0$ be fixed, $\alpha \in \left[\frac{1}{2}, 1\right)$, $\beta \in \left(0, \frac{1}{2}\right]$, $\alpha + \beta = 1$, and $\operatorname{div} u_0 = 0$. Let $m > 2$ be an integer, and $(u_0, \theta_0) \in H^m\left(\mathbb{R}^2\right)$. Then, there exists a unique solution $(u, \theta)$ to the Cauchy problem (3.21) such that

$$\theta \in C([0, \infty); H^m(\mathbb{R}^2)) \cap L^2(0, \infty; H^{m+\beta}(\mathbb{R}^2)),$$

and

$$u \in C([0, \infty); H^m(\mathbb{R}^2)) \cap L^2(0, \infty; H^{m+\alpha}(\mathbb{R}^2)).$$

Remark 1.1. For simplicity of the exposition, we formulate and prove Theorem 3.4 in the subcritical case $\alpha + \beta = 1$, only. One can easily verify that, by arguments from this work, we can obtain an analogous result for $1 \le \alpha + \beta \le 2$.

First of all, let us give a positive inequality in framework of $L^p$.

**Lemma 3.1 (Positive Inequality)**

Let $0 \le \alpha \le 2$. For every $p > 1$, we have

$$\int_{\mathbb{R}^n} (\Lambda^\alpha w) |w|^{p-2} w \, \mathrm{d}x \ge C(p) \int_{\mathbb{R}^n} \left( \Lambda^{\frac{\alpha}{2}} |w|^{\frac{p}{2}} \right)^2 \, \mathrm{d}x, \tag{3.22}$$

for all $w \in L^p(\mathbb{R}^n)$ such that $\Lambda^\alpha w \in L^p(\mathbb{R}^n)$, where $C(p) = \frac{4(p-1)}{p^2}$.

This inequality is well-known in the theory of sub-Markovian operators and its statement and the proof is given e.g. in (V.A. Liskevich, Yu.A. Semenov, Some problems on Markov semigroups, Schrödinger operators, Markov semigroups, wavelet analysis, operator algebras, 1996,). Observe that if $\alpha = 2$, integrating by parts we obtain (3.22) with the equality.

**Theorem 3.5 (Blow-up Criterion Xiaojing Xu 2010)**

Let $\alpha, \beta \in [0, 2]$, $\nu \geq 0$, $\kappa \geq 0$. Suppose $(u_0, \theta_0) \in H^m(\mathbb{R}^2)$ with $m > 2$ being an integer. Then, there exists a unique local classical solution $(u, \theta) \in C([0, T); H^m(\mathbb{R}^2))$ of problem (3.21) for some $T = T(\|u_0\|_{H^m(\mathbb{R}^2)}, \|\theta_0\|_{H^m(\mathbb{R}^2)})$. Moreover, the solution remains in $H^m\left(\mathbb{R}^2\right)$ up to a time $T_1 > T$, namely $(u, \theta) \in C\left([0, T_1]; H^m\left(\mathbb{R}^2\right)\right)$ if and only if

$$\int_0^T \|\nabla\theta(\tau)\|_{L^\infty} \mathrm{d}\tau < \infty. \tag{3.23}$$

• In the inviscid case $\nu = 0$ and $\kappa = 0$, Blow up Criterion was proved in Chae1997. The arguments from that works with minor changes also for problem (3.21) with the fractional diffusion, due to inequality (3.22). By this reason, we skip details of the proof of Theorem 3.5.

• In order to prove Theorem 3.4, it suffices to show that (3.23) holds true for the smooth solutions $(u, \theta)$ to the Cauchy problem (3.21).

• In the following section, we first show some a priori estimates for a smooth solution $(u, \theta) \in C([0, T); H^m(\mathbb{R}^2))$ with $m > 2$ to (3.21), then prove that (3.23) is valid.

For simplicity, let $\nu = \kappa = 1$.

Let $p \geq 2$. Multiplying the second equation in (3.21) by $|\theta|^{p-2}\theta$ and integrating over $\mathbb{R}^2$, we deduce that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta(t)\|_{L^p}^p + \int_0^t (-\Delta)^\beta \theta |\theta|^{p-2}\theta \mathrm{d}x = 0,$$

where we have used the divergence free condition. This identity together with Lemma 3.1, allows us to get

$$\|\theta(t)\|_{L^p}^p + C(p)\int_0^t \|\Lambda^\beta |\theta|^{\frac{p}{2}}(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \leq \|\theta_0\|_{L^p}^p.$$

In particular, when $p = 2$, we have

$$\|\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\beta \theta(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \leq \|\theta_0\|_{L^2}^2. \tag{3.24}$$

- Estimate of $\|u\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))}$.

Multiplying the first equation of (3.21) by $u$, and integrating it over $\mathbb{R}^2$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2}^2 + \int_{\mathbb{R}^2} u(-\Delta)^\alpha u \,\mathrm{d}x = \int_{\mathbb{R}^2} \theta e_2|u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^2}(u\cdot\nabla)|u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^2} \nabla P u \,\mathrm{d}x.$$

This identity together with inequality (3.22) and the divergence free condition, yield that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2}^2 + \int_{\mathbb{R}^2} |\Lambda^\alpha u|^2 \,\mathrm{d}x = \int_{\mathbb{R}^2} \theta e_2 u \,\mathrm{d}x \le \|\theta(t)\|_{L^2}\|u(t)\|_{L^2}.$$

By (3.24) and the Hölder inequality, we deduce that

$$\|u(t)\|_{L^2}^2 + 4\int_0^t \|\Lambda^\alpha u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \le 4\|\theta_0\|_{L^2}^2 T^2 + 2\|u_0\|_{L^2}^2.$$

- Estimate of $\|\omega(t)\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))}$.

Taking the operation curl on both sides of the first equation in (3.21) and denoting $\omega = \operatorname{curl} u = \partial_{x_1} u_2 - \partial_{x_2} u_1$, we get

$$\omega_t + (-\Delta)^\alpha \omega + (u \cdot \nabla)\omega = -\theta_{x_1}. \tag{3.25}$$

Multiplying the above equality by $\omega$, integrating over $\mathbb{R}^2$, we find

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t)\|_{L^2}^2 + \|\Lambda^\alpha \omega\|_{L^2}^2 &= \frac{1}{2}\int_{\mathbb{R}^2} (u\cdot\nabla)|\omega|^2 \,\mathrm{d}x - \int_{\mathbb{R}^2} \theta_{x_1}\omega \mathrm{d}x \\
&= -\int_{\mathbb{R}^2} \theta_{x_1}\omega \mathrm{d}x \\
&\le \frac{1}{2}\|\Lambda^\alpha \omega\|_{L^2}^2 + \frac{1}{2}\|\Lambda^\beta \theta\|_{L^2}^2
\end{aligned}$$

Here, in the last inequality, we have used the Parseval theorem and the relation $\alpha + \beta = 1$.

Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t)\|_{L^2}^2 + \|\Lambda^\alpha \omega\|_{L^2}^2 \le \|\Lambda^\beta \theta\|_{L^2}^2.$$

By virtue of estimate (3.24), we deduce that

$$\begin{aligned}\|\omega(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\alpha \omega(\tau)\|_{L^2}^2 \ \mathrm{d}\tau &\le \|\omega_0\|_{L^2}^2 + \int_0^t \|\Lambda^\beta \theta(\tau)\|_{L^2}^2 \ \mathrm{d}\tau \\ &\le C\left(\|\omega_0\|_{L^2}, \|\theta_0\|_{L^2}, T\right).\end{aligned} \tag{3.26}$$

- Estimate of $\|\Lambda^\alpha D\omega\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))}$

We first compute the derivative $\nabla = (\partial_{x_1}, \partial_{x_2})$ of both sides of (3.25), and then take $L^2$ inner product with $\nabla\omega$. After integration by parts, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\omega(t)\|_{L^2}^2 + \|\Lambda^\alpha\nabla\omega\|_{L^2}^2 &= -\int_{\mathbb{R}^2}[\nabla(u\cdot\nabla)\omega]\nabla\omega \mathrm{d}x - \int_{\mathbb{R}^2}\nabla\theta_{x_1}\nabla\omega \mathrm{d}x \\
&= -\int_{\mathbb{R}^2}[(\nabla u\cdot\nabla)\omega]\nabla\omega \mathrm{d}x - \int_{\mathbb{R}^2}\nabla\theta_{x_1}\nabla\omega \mathrm{d}x \\
&\le \|\nabla u\|_{L^{\frac{2}{1-\alpha}}}\|\nabla\omega\|_{L^2}\|\nabla\omega\|_{L^{\frac{2}{\alpha}}} + \frac{1}{2}\|\Lambda^\alpha\nabla\omega\|_{L^2}^2 + \frac{1}{2}\|\Lambda^\beta\nabla\theta\|_{L^2}^2 \\
&\le \|\nabla u\|_{\frac{2}{1-\alpha}}\|\nabla\omega\|_{L^2}^{\frac{3\alpha-1}{\alpha}}\|\Lambda^\alpha\nabla\omega\|_{L^2}^{\frac{1-\alpha}{\alpha}} + \frac{1}{2}\|\Lambda^\alpha\nabla\omega\|_{L^2}^2 + \frac{1}{2}\|\Lambda^\beta\nabla\theta\|_{L^2}^2 \\
&\le C\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}^{\frac{2\alpha}{3\alpha-1}}\|\nabla\omega\|_{L^2}^2 + \frac{3}{4}\|\Lambda^\alpha\nabla\omega\|_{L^2}^2 + \frac{1}{2}\|\Lambda^\beta\nabla\theta\|_{L^2}^2
\end{aligned} \tag{3.27}$$

where we have used the assumptions $\alpha \ge \frac{1}{2}$ and $\operatorname{div} u = 0$.

Next, computing the derivative $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$ of the second equation from (3.21), we easily show that

$$\nabla^\perp \theta_t + \nabla^\perp[(u\cdot\nabla)\theta] + (-\Delta)^\beta \nabla^\perp \theta = 0.$$

We multiply the above equality by $\nabla^\perp\theta$, and integrate it over $\mathbb{R}^2$. Similar arguments as those in (3.27) lead to

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^\perp\theta(t)\|_{L^2}^2 + \|\Lambda^\beta\nabla^\perp\theta\|_{L^2}^2 \leq & -\int_{\mathbb{R}^2}(u\cdot\nabla)\nabla^\perp\theta\nabla^\perp\theta\mathrm{d}x \\ & -\int_{\mathbb{R}^2}\nabla^\perp\theta\cdot\nabla u\nabla^\perp\theta\mathrm{d}x \\ =&\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}\|\nabla^\perp\theta\|_{L^2}\|\nabla^\perp\theta\|_{L^{\frac{2}{\alpha}}} \\ \leq&\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}\|\nabla^\perp\theta\|_{L^2}\|\Lambda^\beta\nabla^\perp\theta\|_{L^2} \\ \leq&\frac{1}{2}\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}^2\|\nabla^\perp\theta\|_{L^2}^2 + \frac{1}{2}\|\Lambda^\beta\nabla^\perp\theta\|_{L^2}^2.\end{aligned} \tag{3.28}$$

Now, combining (3.27) with (3.28), one can show the function

$$X(t) = \|\nabla\omega(t)\|_{L^2} + \|\nabla^{\perp}\theta(t)\|_{L^2}.$$

satisfies the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) \le C(\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}^{\frac{2\alpha}{3\alpha-1}} + \|\nabla u\|_{L^{\frac{2}{1-\alpha}}}^{2})X(t).$$

Therefore, Gronwall's inequality and the embedding inequality and estimate (3.26) yield that

$$\begin{aligned}
X(t) &\le CX(0)\exp\{\int_0^t (\|\nabla u(\tau)\|_{L^{\frac{2}{1-\alpha}}}^{\frac{2\alpha}{3\alpha-1}} + \|\nabla u(\tau)\|_{L^{\frac{2}{1-\alpha}}}^{2})\mathrm{d}\tau\} \\
&\le CX(0)\exp\{\int_0^t (\|\Lambda^{\alpha}\omega(\tau)\|_{L^2}^{\frac{2\alpha}{3\alpha-1}} + \|\Lambda^{\alpha}\omega(\tau)\|_{L^2}^{2})\mathrm{d}\tau\} \\
&\le CX(0)\exp\left\{T^{\frac{4\alpha-2}{3\alpha-1}}\left(\int_0^t \|\Lambda^{\alpha}\omega(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{\frac{2\alpha}{3\alpha-1}} + \int_0^t \|\Lambda^{\alpha}\omega(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right\} \\
&\le C(T, \|u_0\|_{H^2}, \|\theta_0\|_{H^1}).
\end{aligned}$$

Finally, by virtue of estimate (3.27), we deduce

$$\|\nabla\omega(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\alpha \nabla\omega(\tau)\|_{L^2}^2 \, \mathrm{d}\tau \leq C\left(T, \|u_0\|_{H^2}, \|\theta_0\|_{H^1}\right).$$

- Estimate of $\|\nabla\theta\|_{L^\infty(0,\infty;L^\infty(\mathbb{R}^2))}$

Multiplying (3.9) by $\left|\nabla^\perp\theta\right|^{p-2}\nabla^\perp\theta$, and integrating it over $\mathbb{R}^2$, we have

$$\begin{aligned}\frac{1}{p}\frac{d}{\mathrm{d}t}\|\nabla^\perp\theta(t)\|_{L^p}^p + C(p)\|\nabla^\perp\theta\|^p_{L^{\frac{2p}{1-\alpha}}} &\le \int_{\mathbb{R}^2}\nabla^\perp\theta\cdot\nabla u|\nabla^\perp\theta|^{p-2}\nabla^\perp\theta\mathrm{d}x\\ &\le \|\nabla u\|_{L^\infty}\|\nabla^\perp\theta\|_{L^p}^p.\end{aligned}$$

Using Gronwall's inequality and the obvious identity $\|\nabla Q\|_{L^p} = \|\nabla^\perp Q\|_{L^p}$, we easily show that

$$\|\nabla\theta(t)\|_{L^p} \le C\,\|\nabla\theta_0\|_{L^p}\exp\left\{\int_0^t\|\nabla u(\tau)\|_{L^\infty}\mathrm{d}\tau\right\}.$$

This inequality together with the Gagliardo-Nirenberg inequality, allows us to obtain

$$\begin{aligned}\int_0^t \|\nabla u(\tau)\|_{L^\infty} \mathrm{d}\tau &\le C \int_0^t \|\omega(\tau)\|_{L^2}^{\frac{\alpha}{1+\alpha}} \left\|\Lambda^{1+\alpha}\omega(\tau)\right\|_{L^2}^{\frac{1}{1+\alpha}} \mathrm{d}\tau \\ &\le CT^{\frac{2\alpha+1}{\alpha+1}} \left(\int_0^t \|\omega(\tau)\|_{L^2}^2 \,\mathrm{d}\tau\right)^{\frac{\alpha}{1+2\alpha}} + C\int_0^t \left\|\Lambda^{1+\alpha}\omega(\tau)\right\|_{L^2}^2 \,\mathrm{d}\tau \\ &\le C\left(T, \|u_0\|_{H^2}, \|\theta_0\|_{H^1}\right).\end{aligned}$$

Using Sobolev embedding

$$\|f(x)\|_{L^p(\mathbb{R}^2)} \le C\|f(x)\|_{H^s(\mathbb{R}^2)}, \quad s > 1,$$

where $C$ is independent of $p \in [2, \infty]$, then we have

$$\|\nabla\theta(t)\|_{L^p} \le C\|\theta_0\|_{H^m} \exp\left\{\int_0^t \|\nabla u(\tau)\|_{L^\infty} \mathrm{d}\tau\right\} \le C \quad \text{for all } t \in [0, T]$$

where $C$ is independent of $p$.

Passing to the limit $p \to \infty$ in above inequality, we obtain

$$\|\nabla\theta(t)\|_{L^\infty} \leq C\left(T, \|u_0\|_{H^2}, \|\theta_0\|_{H^m}\right), \quad \forall t \in [0, T].$$

This implies that condition (3.23) holds true, and according to Theorem 3.5, we obtain a unique solution of (3.21) such that $(u, \theta) \in C\left([0, \infty); H^m\left(\mathbb{R}^2\right)\right)$. By (3.27), (3.28) and the iteration process, we construct $u \in L^2\left(0, \infty; H^{m+\alpha}\left(\mathbb{R}^2\right)\right)$ and $\theta \in L^2\left(0, \infty; H^{m+\beta}\left(\mathbb{R}^2\right)\right)$, and we complete the proof of Theorem 3.4.
$\square$

# The Critical Dissipation