Theory on Well-posedness of Boussinesq equations

4. Asymptotic behavior of solution

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4.2 Vanishing Dissipative Limit

i. Full space $\mathbb{R}^2$
ii. Domain with boundary

The 2D Boussineq system with full dissipation is as follows:

$$(B)\begin{cases} v_t + (v\cdot\nabla)v = -\nabla p + \nu\Delta v + \theta e_2, \quad (x,\,t)\in\mathbb{R}^2\times\mathbb{R}_+,\\ \theta_t + (v\cdot\nabla)\theta = \kappa\Delta\theta\\ \operatorname{div} v = 0\\ v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases}$$

Boussinesq system with the zero viscosity is as follows:

$$(B_2)\begin{cases} v_t + (v\cdot\nabla)v = -\nabla p + \theta e_2, \quad (x,\,t)\in\mathbb{R}^2\times\mathbb{R}_+,\\ \theta_t + (v\cdot\nabla)\theta = \kappa\Delta\theta\\ \operatorname{div} v = 0\\ v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x) \end{cases}$$

Let (v, p, θ) and $(\tilde{v}, \tilde{p}, \tilde{\theta})$ be solutions of (B_2) and (B), respectively with the same initial conditions (v_0, θ_0). Similar to the global well-posedness of partial dissipation case, we can get the key ν-independent estimate for the solutions $(\tilde{v}, \tilde{\theta})$ is

$$\begin{aligned} &\|\nabla\tilde{v}\|_{L^\infty} + \|\nabla\tilde{\theta}\|_{L^\infty} + \|\tilde{v}\|_{W^{2,p}} + \|\tilde{\theta}\|_{W^{2,p}} \\ \leqslant & C\left(\|v_0\|_{W^{2,p}}, \|\theta_0\|_{W^{2,p}}, \kappa, T, p\right). \end{aligned} \tag{1}$$

From $(B)-(B_2)$ we obtain for $\Theta=\theta-\tilde{\theta}$, $P=p-\tilde{p}$, $V=v-\tilde{v}$

$$\Theta_t+(v\cdot\nabla)\Theta+(V\cdot\nabla)\tilde{\theta}=\kappa\Delta\Theta. \tag{2}$$

with $\operatorname{div} V=0$, and

$$V_t+(v\cdot\nabla)V+(V\cdot\nabla)\tilde{v}=-\nabla P+\Theta e_2+\nu\Delta V+\nu\Delta\tilde{v}. \tag{3}$$

Taking L^2 inner product (2) with Θ, after integration by part we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Theta\|_{L^2}^2+\kappa\|\nabla\Theta\|_{L^2}^2&=-\int_{\mathbb{R}^2}(V\cdot\nabla)\tilde{\theta}\Theta dx\\&\leqslant\|\nabla\tilde{\theta}\|_{L^\infty}\|V\|_{L^2}\|\Theta\|_{L^2}\\&\leqslant C\|V\|_{L^2}^2+C\|\Theta\|_{L^2}^2,\end{aligned}$$

where $C=C(v_0,\theta_0,T,\kappa)$, and we have used estimate (1).

Hence,

$$\frac{d}{dt}\|\Theta\|_{L^2}^2 \leqslant C\|\Theta\|_{L^2}^2 + C\|V\|_{L^2}^2. \tag{4}$$

Take L^2 inner product (3) with V, we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|V\|_{L^2}^2 + \nu\|\nabla V\|_{L^2}^2 \\
= & -\int_{\mathbb{R}^2}(V\cdot\nabla)\tilde{v}\cdot V dx + \int_{\mathbb{R}^2}\Theta\, e_2\cdot V dx - \nu\int_{\mathbb{R}^2}\nabla\tilde{v}\cdot\nabla V dx \\
\leqslant & \|\nabla\tilde{v}\|_{L^\infty}\|V\|_{L^2}^2 + \|\Theta\|_{L^2}\|V\|_{L^2} + \nu\|\nabla\tilde{v}\|_{L^2}\|\nabla V\|_{L^2} \\
\leqslant & C\left(\|V\|_{L^2}^2 + \|\Theta\|_{L^2}^2\right) + \frac{\nu}{2}\|\nabla V\|_{L^2}^2 + \frac{\nu}{2}\|\nabla\tilde{v}\|_{L^2}^2,
\end{aligned}$$

where $C = C(v_0, \theta_0, T, \kappa)$, and used estimate (1) again. Then we obtain that

$$\frac{d}{dt}\|V\|_{L^2}^2 \leqslant C\left(\|V\|_{L^2}^2 + \|\Theta\|_{L^2}^2\right) + \frac{\nu}{2}\|\nabla\tilde{v}\|_{L^2}^2. \tag{5}$$

Adding (5) to (4), and setting $X(t) = \|\Theta(t)\|_{L^2}^2 + \|V(t)\|_{L^2}^2$, we obtain that

$$\frac{d}{dt}X(t) \leqslant CX(t) + C\nu\|\nabla\tilde{v}\|_{L^2}^2.$$

By Gronwall's lemma we find that

$$\begin{aligned} X(t) &\leqslant X(0)e^{Ct} + C\nu\int_0^t \|\nabla\tilde{v}(s)\|_{L^2}^2 e^{C(t-s)}ds \\ &\leqslant Ce^{CT}\nu\int_0^T \|\nabla\tilde{v}(t)\|_{L^2}^2 dt \leqslant C\nu \end{aligned}$$

where we used (1) and the fact that $X(0) = 0$.

Hence, we obtain

$$\sup_{0\leqslant t\leqslant T}\left(\|v(t)-\tilde{v}(t)\|_{L^2}+\|\theta(t)-\tilde{\theta}(t)\|_{L^2}\right)\leqslant C\sqrt{\nu},$$

where $C=C(v_0,\theta_0,T,\kappa)$. Then we have, for $0\leqslant s<m$

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}\|v(t)-\tilde{v}(t)\|_{H^s}&\leqslant C\sup_{0\leqslant t\leqslant T}\|v(t)-\tilde{v}(t)\|_{L^2}^{\sigma}\|v(t)-\tilde{v}(t)\|_{H^m}^{1-\sigma}\\&\leqslant C\left(\|v_0\|_{H^m}+\|\tilde{v}_0\|_{H^m}\right)^{1-\sigma}\sup_{0\leqslant t\leqslant T}\|v(t)-\tilde{v}(t)\|_{L^2}^{\sigma}\\&\leqslant C\nu^{\frac{m-s}{2m}},\end{aligned}$$

where $\sigma=1-\frac{s}{m}$ and $C=C(v_0,\theta_0,T,\kappa,s,m)$, and similarly for $\|\theta-\tilde{\theta}\|_{H^s}$, we obtain the desired convergence $(\tilde{v},\tilde{\theta})\to(v,\theta)$ in $C\left([0,T];H^s\left(\left(\mathbb{R}^2\right)\right)\right)$ as $\nu\to 0$.

Introduction–Boussinesq system

2-D Boussinesq system:

$$\begin{cases} \partial_t u^{\nu,\kappa} + u^{\nu,\kappa} \cdot \nabla u^{\nu,\kappa} + \nabla p^{\nu,\kappa} = \nu \Delta u^{\nu,\kappa} + \theta^{\nu,\kappa} e_2, & t > 0,\ (x,y) \in \Omega, \\ \partial_t \theta^{\nu,\kappa} + u^{\nu,\kappa} \cdot \nabla \theta^{\nu,\kappa} = \kappa \Delta \theta^{\nu,\kappa}, \\ \nabla \cdot u^{\nu,\kappa} = 0, \\ (u^{\nu,\kappa}, \theta^{\nu,\kappa})|_{t=0} = (u_0, \theta_0). \end{cases} \tag{6}$$

- $u^{\nu,\kappa} = (u_1^{\nu,\kappa}(t;x,y), u_2^{\nu,\kappa}(t;x,y))$ is velocity.
- $\theta^{\nu,\kappa} = \theta^{\nu,\kappa}(t;x,y)$ is temperature.
- $p^{\nu,\kappa} = p^{\nu,\kappa}(t;x,y)$ is pressure.

· $\Omega \subset \mathbb{R}^2$ has smooth boundary.

· $\nu \geq 0$ is fluid viscosity.

· $\kappa \geq 0$ is thermal diffusivity.

· $\theta^{\nu,\kappa} e_2$ represents the buoyancy force, $e_2 = (0, 1)$.

Introduction–Boussinesq system

Two types of boundary conditions:

- Dirichlet boundary conditions

$$\begin{cases} u^{\nu,\kappa}|_{\partial\Omega} = 0, \\ \theta^{\nu,\kappa}|_{\partial\Omega} = \underline{\theta}(t;x,y). \end{cases} \tag{7}$$

- Navier-slip and Neumann boundary conditions

$$\begin{cases} u^{\nu,\kappa}\cdot n|_{\partial\Omega} = 0, \quad (D(u^{\nu,\kappa})n + au^{\nu,\kappa})\cdot\tau|_{\partial\Omega} = 0, \\ \dfrac{\partial\theta^{\nu,\kappa}}{\partial n}|_{\partial\Omega} = 0. \end{cases} \tag{8}$$

· $D_{ij}(u^{\nu,\kappa}) = \frac{\partial_i u_j^{\nu,\kappa} + \partial_j u_i^{\nu,\kappa}}{2}$.
· n is the unit normal vector at $\partial\Omega$.
· τ is the tangent unit vector at $\partial\Omega$.
· a is the scalar friction function.

Introduction–non-dissipative Boussinesq system

The limit case $\nu = \kappa = 0$:

$$\begin{cases} \partial_t u^0 + u^0 \cdot \nabla u^0 + \nabla p^0 = \theta^0 e_2, \quad t > 0,\, (x,y) \in \Omega, \\ \partial_t \theta^0 + u^0 \cdot \nabla \theta^0 = 0, \\ \nabla \cdot u^0 = 0, \\ (u^0, \theta^0)|_{t=0} = (u_0, \theta_0), \end{cases} \tag{9}$$

with boundary conditions

$$u^0 \cdot \mathbf{n}\,|_{\partial\Omega} = 0. \tag{10}$$

Question: In what scene $(u^{\nu,\kappa}, \theta^{\nu,\kappa}) \to (u^0, \theta^0)$?
Difficulty: Boundary layer always occurs.

Results of vanishing viscosity/diffusivity limit

- Jiang-Zhang-Zhao, (J. Differ. Equ., 2011).
  (6) (7) with $\nu > 0$, $\kappa \to 0$ in $\Omega = \mathbb{R}^2_+$.
- Fan-Xu-Wang-Wang, (Appl. Anal., 2020)
  (6) (7) with $\nu > 0$, $\kappa \to 0$ in a 3D channel.
- Wang-Xie, (Proc. Roy. Soc. Edinburgh Sect., 2015).
  (6) (8) with $\kappa = h\nu \to 0$ in bounded domain Ω.
- Gie-Whitehead, (J. Math. Fluid Mech., 2019).
  (6) (8) with $\kappa \to 0$ or $\nu \to 0$ in a 3D channel domain.

Introduction–boundary layer theory

Prandtl's theory in 1904

- The asymptotic limit of the fluid flow as viscosity tends to zero, $\Omega = \{(x,y); 0 < x < X, 0 < y < +\infty\}$.

$$\begin{cases} \partial_t u^\nu + u^\nu \cdot \nabla u^\nu + \nabla p^\nu = \nu \Delta u^\nu, \quad t > 0,\ (x,y) \in \Omega, \\ \nabla \cdot u^\nu = 0, \\ u^\nu|_{y=0} = 0, \\ u^\nu|_{t=0} = u_0. \end{cases} \tag{NS}$$

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, \quad t > 0,\ (x,y) \in \Omega, \\ \nabla \cdot u = 0, \\ u \cdot n|_{y=0} = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{Euler}$$

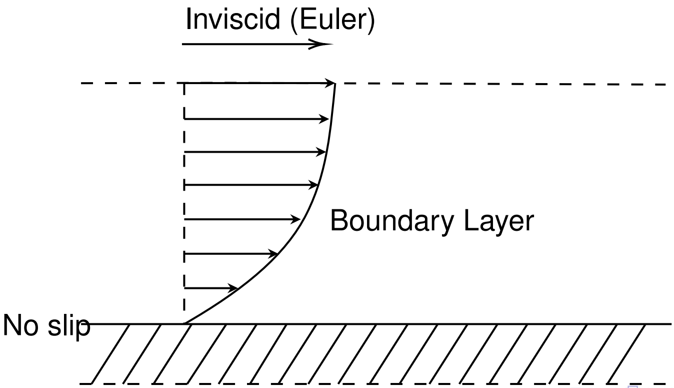
Whether

$$u^\nu \to u, \quad \text{as } \nu \to 0?$$

Introduction–boundary layer theory

- Convergence of solutions of (NS) to solutions of (Euler) and Prandtl's equations.

$$u^{\nu} \approx u + u_{BL}.$$



Results of vanishing viscosity limit

Validity of

$$\|u^\nu - u\|_{L^\infty(0,T;L^2(\Omega))} \to 0, \quad \text{as } \nu \to 0. \tag{11}$$

- Kato, (Math. Sci. Res. Inst. Publ., 1984).
  $\nu \int_0^T \int_\Omega |\nabla u^\nu|^2 dxdt$, or $\nu \int_0^T \int_{\Omega_\nu} |\nabla u^\nu|^2 dxdt \to 0 \Leftrightarrow$ (11).
- Temam-Wang, (Ann. Sc. Norm. Sup. Pisa Cl. Sci., 1997).
  Ω is a channel, $\|p^\nu\|_{L^2(0,T;H^{1/2}(\partial\Omega))}$ bounded,
  or $\nu^{\frac{1}{2}}\|\partial_x u_1^\nu\|_{L^2(0,T;L^2(\Omega_\nu))} \le \nu^{1-\delta}$, $\frac{3}{4} \le \delta < 1$, $\Rightarrow$ (11).
- Constantin-Kukavica-Vicol, (Proc. Amer. Math. Soc., 2015).
  $\Omega = \mathbb{R}^2_+$, $u_1|_{\partial\mathbb{R}^2_+} \ge 0$ and $\omega|_{\partial\mathbb{R}^2_+} = (\partial_1 u_2^\nu - \partial_2 u_1^\nu)|_{\partial\mathbb{R}^2_+} \ge 0 \Rightarrow$ (11).
- Masmoudi, (Arch. Ration. Mech. Anal., 1998).
  $\Omega = \mathbb{R}^3_+$, $\nu\Delta$ is replaced by $\eta\Delta_{x,y} + \nu\partial_z^2$. $\nu, \eta, \frac{\nu}{\eta} \to 0 \Rightarrow$ (11).

Other results of vanishing viscosity limit

For Navier slip boundary condition:

- Iftimie-Sueur, (Arch. Ration. Mech. Anal., 2011).
- Gie-Kelliher, (J. Differ. Equ., 2012).
- Tao-Wang-Zhang, (SAIM J. Math. Anal., 2020).

For anisotropic NS equations:

- Wang-Wang-Xin, (Commun. Math. Sci., 2010).
  $\epsilon^{\frac{1}{2}}\partial_x^2 u^\epsilon + \epsilon\partial_y^2 u^\epsilon$, $\beta u_1 - \epsilon^{\frac{1}{4}}\frac{\partial u_1^\epsilon}{\partial y}|_{y=0} = 0$, $u_2^\epsilon|_{y=0} = 0$, with $\epsilon \to 0$.
- Liu-Wang, (Z. Angew. Math. Phs., 2013).
  $\nu(\partial_{x_1}^2 + \partial_{x_2}^2)u + \epsilon\partial_{x_3}^2 u$, with $\nu > 0$, $\epsilon \to 0$.

Introduction–Anisotropic Boussinesq equations, Dirichlet boundary conditions

Consider

$$\begin{cases} \partial_t u^\epsilon + u^\epsilon \cdot \nabla u^\epsilon + \nabla p^\epsilon = \epsilon_1 \partial_{xx} u^\epsilon + \epsilon_2 \partial_{yy} u^\epsilon + \theta^\epsilon e_2,\ t > 0,\ (x,y) \in \mathbb{R}^2_+ \\ \partial_t \theta^\epsilon + u^\epsilon \cdot \nabla \theta^\epsilon = \epsilon_3 \partial_{xx} \theta^\epsilon + \epsilon_4 \partial_{yy} \theta^\epsilon, \\ \nabla \cdot u^\epsilon = 0, \\ u^\epsilon|_{y=0} = 0, \quad \theta^\epsilon|_{y=0} = a(t;x), \\ (u^\epsilon, \theta^\epsilon)|_{t=0} = (u_0, \theta_0), \end{cases} \tag{12}$$

where $\epsilon_i \to 0, i = 1, \cdots, 4.$

- A natural problem: $(u^\epsilon, \theta^\epsilon) \to (u^0, \theta^0)$, as $\epsilon_i \to 0$?
- Difficulties arise: $(u^\epsilon, \theta^\epsilon)|_{y=0} \neq (u^0, \theta^0)|_{y=0}$.
  - $\not\Rightarrow \|(u^\epsilon, \theta^\epsilon) - (u^0, \theta^0)\|_{H^1(\mathbb{R}^2_+)}$.
  - Boundary layer exists!

Our aim

- When ϵ_i goes to 0 with different speed, how does ϵ_i affect the vanishing dissipation limit?

$$\epsilon_i = \epsilon^{\lambda_i}, \ \lambda_i > 0, \ i = 1, \cdots, 4.$$

- Establishing $\|(u^\epsilon, \theta^\epsilon) - (u^0, \theta^0)\|_{L^\infty(0,T;L^2(\mathbb{R}^2_+))} \to 0$, as $\epsilon \to 0$.
- Calculating the convergence rate.

Global well-posedness result

Theorem 1

Assume $u_0 \in H^2(\mathbb{R}^2_+)$, $\theta_0 \in H^1(\mathbb{R}^2_+) \cap W^{1,\infty}(\mathbb{R}^2_+)$, $a \in L^\infty(0,+\infty; H^3(\mathbb{R}))$, $a_t \in L^\infty(0,+\infty; H^1(\mathbb{R}))$, *and* (u_0, θ_0) *satisfies the corresponding compatibility conditions. Then for* $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 > 0$, (12) *admits a unique global strong solution* $(u^\epsilon, \theta^\epsilon)$, *such that for any given* $T > 0$,

$$u^\epsilon \in L^\infty(0,T; H^1_0(\mathbb{R}^2_+) \cap H^2(\mathbb{R}^2_+)),$$

$$\theta^\epsilon \in L^\infty(0,T; H^1(\mathbb{R}^2_+)) \cap L^2(0,T; H^2(\mathbb{R}^2_+)).$$

- Jiang-Zhang-Zhao, JDE, (2011).

Local well-posedness result

Theorem 2 (Peixin Wang,X., IUMJ, 2021)

Assume $(u_0, \theta_0) \in H^s(\mathbb{R}^2_+)$, *with* $s > 2$, *integer. Then* (9) (10) *have a unique solution* (u^0, θ^0) *locally in time. That is, there exists* $T_1 > 0$ *such that*

$$u^0 \in L^\infty(0, T_1; H^s(\mathbb{R}^2_+)),$$

$$\theta^0 \in L^\infty(0, T_1; H^s(\mathbb{R}^2_+)).$$

Moreover,

$$\|u^0\|_{L^\infty(0,T_1;H^s(\mathbb{R}^2_+))} \leq C(u_0, \theta_0),$$

$$\|\theta^0\|_{L^\infty(0,T_1;H^s(\mathbb{R}^2_+))} \leq C(u_0, \theta_0).$$

Vanishing dissipation result

Theorem 3 (Peixin Wang, X., IUMJ, 2021)

Assume $(u_0, \theta_0) \in H^s(\mathbb{R}^2_+)$ *satisfies the corresponding compatibility conditions, with* $s > 2$*, integer.* $(u^\epsilon, \theta^\epsilon)$ *is the solution of* (12)*, and* (u^0, θ^0) *is the solution of* (9) (10)*. Let* $\lambda_1 < \min\{\lambda_2,\ \lambda_4\}$, $\lambda_1 < 4\lambda_2 - \lambda_4$*. Then, there exists* $\alpha \in (\frac{\lambda_1+\lambda_4}{4}, \lambda_2)$*, such that*

$$\|(u^\epsilon, \theta^\epsilon) - (u^0, \theta^0)\|_{L^\infty(0,T_1;L^2(\mathbb{R}^2_+))} \le C(T_1, u_0, \theta_0, a)\delta,$$

where the constant C *is independent of* ϵ, ϵ_i $(i = 1, \cdots, 4)$*, and*

$$\delta = \epsilon^{\frac{1}{2}\min\{\lambda_1,\ \lambda_2-\alpha,\ \lambda_3,\ (\frac{\lambda_4-\lambda_1}{2})^-,\ (2\alpha-\frac{\lambda_1+\lambda_4}{2})^-\}}.$$

Compatibility conditions

Since $(u^\epsilon, \theta^\epsilon)|_{t=0} = (u^0, \theta^0)|_{t=0} = (u_0, \theta_0)$, (u_0, θ_0) should satisfy, $\forall \epsilon_2, \epsilon_3, \epsilon_4 > 0$,

$$\begin{cases} u_0|_{y=0} = 0, \quad \nabla \cdot u_0 = 0, \\ \epsilon_2 \partial_{yy} u_0|_{y=0} - \nabla p_0^\epsilon|_{y=0} = -a(0;x)e_2, \\ \theta_0|_{y=0} = a(0;x), \\ \partial_t a(t;x)|_{t=0} = \epsilon_3 \partial_{xx} \theta_0|_{y=0} + \epsilon_4 \partial_{yy} \theta_0|_{y=0}, \end{cases} \tag{13}$$

where p_0^ϵ solves the Neumann boundary problem

$$\begin{cases} \Delta p_0^\epsilon = \nabla \cdot (\theta_0 \mathbf{e_2} - u_0 \cdot \nabla u_0), \quad (x,y) \in \mathbb{R}^2_+, \\ \nabla p_0^\epsilon \cdot \mathbf{n}|_{y=0} = \epsilon_2 \partial_{yy} u_{2,0}|_{y=0} + a(0;x). \end{cases}$$

Compatibility conditions

Inserting p_0^ϵ into $(13)_2$,

$$\begin{cases} u_0|_{y=0} = 0, \quad \nabla \cdot u_0 = 0, \\ \partial_{yy} u_{1,0}|_{y=0} = \mathcal{H} \partial_{yy} u_{2,0}|_{y=0}, \\ 2 \displaystyle\int_{\mathbb{R}^2_+} \partial_x \Gamma(w - x, z) \nabla \cdot (u_0 \cdot \nabla u_0 - \theta_0 \mathbf{e_2})(w, z) dw dz + \mathcal{H} a(0; x) = 0, \\ \theta_0|_{\partial \mathbb{R}^2_+} = a(0; x), \\ \partial_{xx} \theta_0|_{y=0} = \partial_{yy} \theta_0|_{y=0} = 0, \\ \partial_{xx} a(0; x) = 0. \end{cases}$$

· $\mathcal{H}$ is the Hilbert operator.
· $\Gamma(x, y)$ is the fundamental solution of Laplace's equation.

The sketch of proof:

We show a lemma of Gronwall's type.

Lemma 1 (Masmoudi, ARMA, 1998)

Let $\delta > 0$, *and* f, g, l *be nonnegative functions in* $L^1(0,T)$ *satisfying*

$$\int_0^T f(t)dt \le M,$$

$$\int_0^T g(t)dt \le M\delta, \quad \int_0^T l(t)dt \le M\delta^2.$$

If a nonnegative function F *satisfies*

$$\partial_t(F^2) \le f(t)F^2 + g(t)F + l(t), \quad F(0) \le M\delta,$$

then $|F(t)| \le C(M)\delta$, *for all* $t \ge 0$.

The sketch of proof: boundary layer correctors

Let $\rho \in C^\infty([0,\infty))$, $\rho(0) = 1$, $\int_0^1 \rho(s)ds = 0$, $\mathrm{supp}\rho \subset [0,1]$.
$\alpha, \beta, \gamma > 0$ will be determined.

- Match $u^\epsilon - u^0$ on the boundary.

  Define the boundary corrector by stream function $g^{\epsilon,\alpha}$:

$$\begin{aligned}\phi^{\epsilon,\alpha}(t;x,y) &= (\partial_y g^{\epsilon,\alpha}(t;x,y), -\partial_x g^{\epsilon,\alpha}(t;x,y)) \\ &= \left(-u_1^0(t;x,0)\rho\left(\frac{y}{\epsilon^\alpha}\right), u_{1x}^0(t;x,0)\int_0^y \rho\left(\frac{s}{\epsilon^\alpha}\right)ds\right), \\ g^{\epsilon,\alpha}(t;x,y) &= -u_1^0(t;x,0)\int_0^y \rho\left(\frac{s}{\epsilon^\alpha}\right)ds.\end{aligned}$$

  $\phi^{\epsilon,\alpha}$ satisfies

$$\nabla \cdot \phi^{\epsilon,\alpha}(t;x,y) = 0,$$

$$\phi^{\epsilon,\alpha}(t;x,y) = 0, \quad y > \epsilon^\alpha,$$

$$\phi^{\epsilon,\alpha}(t;x,0) = (-u_1^0(t;x,0), 0) = -u^0(t;x,0).$$

The sketch of proof: boundary layer correctors

$\phi^{\epsilon,\alpha}$ has some vanishing norms.

$$
\begin{aligned}
&\|\phi_1^{\epsilon,\alpha}(y)\|_{L^2_y} \le C(\epsilon^\alpha)^{\frac{1}{2}},\\
&\|\phi_2^{\epsilon,\alpha}(y)\|_{L^2_y} \le C(\epsilon^\alpha)^{\frac{3}{2}},\\
&\|\phi_2^{\epsilon,\alpha}(y)\|_{L^\infty_y} \le C\epsilon^\alpha,\\
&\|y\partial_y\phi_1^{\epsilon,\alpha}(y)\|_{L^2_y} \le C(\epsilon^\alpha)^{\frac{1}{2}},\\
&\|y^2\partial_y\phi_1^{\epsilon,\alpha}(y)\|_{L^\infty_y} \le C\epsilon^\alpha,\\
&\|\partial_y\phi_1^{\epsilon,\alpha}(y)\|_{L^2_y} \le C(\epsilon^\alpha)^{-\frac{1}{2}}.
\end{aligned}
$$

The sketch of proof: boundary layer correctors

- Search a zero boundary condition for θ^ϵ:

$$(\theta^\epsilon - a(t;x)\rho(\frac{y}{\epsilon^\gamma}))|_{y=0} = 0.$$

- Match $\theta^\epsilon - a\rho(\frac{y}{\epsilon^\gamma}) - \theta^0$ at the boundary: Define corrector

$$\psi^{\epsilon,\beta}(t;x,y) = -\theta^0(t;x,0)\rho\left(\frac{y}{\epsilon^\beta}\right).$$

The boundary corrector $\psi^{\epsilon,\beta}$ satisfies

$$\psi^{\epsilon,\beta}(t;x,y) = 0, \quad y > \epsilon^\beta,$$
$$\psi^{\epsilon,\beta}(t;x,0) = -\theta^0(t;x,0).$$

$$\|\psi^{\epsilon,\beta}\|_{L_y^2} \le C(\epsilon^\beta)^{\frac{1}{2}},$$
$$\|y\partial_y\psi^{\epsilon,\beta}\|_{L_y^2} \le C(\epsilon^\beta)^{\frac{1}{2}},$$
$$\|y^2\partial_y\psi^{\epsilon,\beta}\|_{L_y^\infty} \le C\epsilon^\beta,$$
$$\|\partial_y\psi^{\epsilon,\beta}\|_{L_y^2} \le C(\epsilon^\beta)^{-\frac{1}{2}}.$$

The sketch of proof: energy estimates

Set $v^\epsilon = u^\epsilon - u^0 - \phi^{\epsilon,\alpha}$, $w^\epsilon = \theta^\epsilon - a(t;x)\rho(\frac{y}{\epsilon^\gamma}) - \theta^0 - \psi^{\epsilon,\beta}$, and $q^\epsilon = p^\epsilon - p^0$.

$$\begin{aligned}
&\partial_t v^\epsilon + \partial_t \phi^{\epsilon,\alpha} + v^\epsilon \cdot \nabla v^\epsilon + u^0 \cdot \nabla v^\epsilon + v^\epsilon \cdot \nabla \phi^{\epsilon,\alpha} + u^0 \cdot \nabla \phi^{\epsilon,\alpha} \\
&+ \phi^{\epsilon,\alpha} \cdot \nabla \phi^{\epsilon,\alpha} + v^\epsilon \cdot \nabla u^0 + \phi^{\epsilon,\alpha} \cdot \nabla v^\epsilon + \phi^{\epsilon,\alpha} \cdot \nabla u^0 + \nabla q^\epsilon \\
=&\epsilon_1 \partial_{xx} v^\epsilon + \epsilon_1 \partial_{xx} u^0 + \epsilon_1 \partial_{xx} \phi^{\epsilon,\alpha} + \epsilon_2 \partial_{yy} v^\epsilon + \epsilon_2 \partial_{yy} u^0 + \epsilon_2 \partial_{yy} \phi^{\epsilon,\alpha} \\
&+ w^\epsilon \mathbf{e_2} + (\psi^{\epsilon,\beta} + a\rho(\frac{y}{\epsilon^\gamma}))\mathbf{e_2},
\end{aligned}$$

$$\begin{aligned}
&\partial_t w^\epsilon + \partial_t \psi^{\epsilon,\beta} + v^\epsilon \cdot \nabla w^\epsilon + v^\epsilon \cdot \nabla \psi^{\epsilon,\beta} + u^0 \cdot \nabla w^\epsilon + u^0 \cdot \nabla \psi^{\epsilon,\beta} \\
&+ \phi^{\epsilon,\alpha} \cdot \nabla w^\epsilon + v^\epsilon \cdot \nabla \theta^0 + \phi^{\epsilon,\alpha} \cdot \nabla \theta^0 + \phi^{\epsilon,\alpha} \cdot \nabla \psi^{\epsilon,\beta} \\
=&\epsilon_3 \partial_{xx} w^\epsilon + \epsilon_3 \partial_{xx} \theta^0 + \epsilon_3 \partial_{xx} \psi^{\epsilon,\beta} + \epsilon_4 \partial_{yy} w^\epsilon + \epsilon_4 \partial_{yy} \theta^0 + \epsilon_4 \partial_{yy} \psi^{\epsilon,\beta} \\
&+ \epsilon_3 \partial_{xx}(a\rho(\frac{y}{\epsilon^\gamma})) + \epsilon_4 \partial_{yy}(a\rho(\frac{y}{\epsilon^\gamma})) + v^\epsilon \cdot \nabla(a\rho(\frac{y}{\epsilon^\gamma})) \\
&- u^0 \cdot \nabla(a\rho(\frac{y}{\epsilon^\gamma})) - \phi^{\epsilon,\alpha} \cdot \nabla(a\rho(\frac{y}{\epsilon^\gamma})) - a_t\rho(\frac{y}{\epsilon^\gamma}),
\end{aligned}$$

The sketch of proof: energy estimates

$$
\begin{aligned}
\nabla \cdot v^\epsilon &= 0, \\
v^\epsilon(t;x,0) &= 0, \\
w^\epsilon(t;x,0) &= 0, \\
v^\epsilon(0;x,y) &= -\phi^{\epsilon,\alpha}(0;x,y), \\
w^\epsilon(0;x,y) &= -a(0;x)\rho(\frac{y}{\epsilon^\gamma}) - \psi^{\epsilon,\beta}(0;x,y).
\end{aligned}
$$

Taking the L^2 norm of v^ϵ, we have that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v^\epsilon\|_{L^2}^2 + \epsilon_1\|\partial_x v^\epsilon\|_{L^2}^2 + \epsilon_2\|\partial_y v^\epsilon\|_{L^2}^2 \\
&= -\int \partial_t \phi^{\epsilon,\alpha} \cdot v^\epsilon - \int v^\epsilon \cdot \nabla \phi^{\epsilon,\alpha} \cdot v^\epsilon - \int u^0 \cdot \nabla \phi^{\epsilon,\alpha} \cdot v^\epsilon - \int \phi^{\epsilon,\alpha} \cdot \nabla \phi^{\epsilon,\alpha} \cdot v^\epsilon \\
&\quad - \int v^\epsilon \cdot \nabla u^0 \cdot v^\epsilon - \int \phi^{\epsilon,\alpha} \cdot \nabla u^0 \cdot v^\epsilon + \epsilon_1 \int \partial_{xx} u^0 \cdot v^\epsilon + \epsilon_1 \int \partial_{xx} \phi^{\epsilon,\alpha} \cdot v^\epsilon \\
&\quad + \epsilon_2 \int \partial_{yy} u^0 \cdot v^\epsilon + \epsilon_2 \int \partial_{yy} \phi^{\epsilon,\alpha} \cdot v^\epsilon + \int w^\epsilon v_2^\epsilon + \int (\psi^{\epsilon,\beta} + a\rho(\frac{y}{\epsilon^\gamma}))v_2^\epsilon.
\end{aligned}
$$

The sketch of proof: energy estimates

$$\left|-\int v^\epsilon \cdot \nabla\phi^{\epsilon,\alpha} \cdot v^\epsilon\right| \le \left|\int v_1^\epsilon \partial_x \phi_1^{\epsilon,\alpha} v_1^\epsilon\right| + \left|\int v_1^\epsilon \partial_x \phi_2^{\epsilon,\alpha} v_2^\epsilon\right| + \left|\int v_2^\epsilon \partial_y \phi^{\epsilon,\alpha} \cdot v^\epsilon\right|.$$

The first one is easy,

$$\left|\int v_1^\epsilon \partial_x \phi_1^{\epsilon,\alpha} v_1^\epsilon\right| \le \|v_1^\epsilon\|_{L^2}^2 \|\partial_x \phi_1^\epsilon\|_{L^\infty} \le C\|u^0\|_{H^s}\|v^\epsilon\|_{L^2}^2.$$

Integrating the second one by parts, we find that

$$\begin{aligned}
\left|\int v_1^\epsilon \partial_x \phi_2^{\epsilon,\alpha} v_2^\epsilon\right| &\le \left|\int \partial_x v_1^\epsilon \phi_2^{\epsilon,\alpha} v_2^\epsilon\right| + \left|\int v_1^\epsilon \phi_2^{\epsilon,\alpha} \partial_x v_2^\epsilon\right| \\
&\le \|\phi_2^{\epsilon,\alpha}\|_{L^\infty}(\|\partial_x v_1^\epsilon\|_{L^2}\|v_2^\epsilon\|_{L^2} + \|v_1^\epsilon\|_{L^2}\|\partial_x v_2^\epsilon\|_{L^2}) \\
&\le C\epsilon^\alpha \|u^0\|_{H^s}\|v^\epsilon\|_{L^2}\|\partial_x v^\epsilon\|_{L^2} \\
&\le \eta_1 \epsilon^{2\alpha}\|\partial_x v^\epsilon\|_{L^2}^2 + \frac{C}{\eta_1}\|u^0\|_{H^s}^2\|v^\epsilon\|_{L^2}^2.
\end{aligned}$$

The sketch of proof: energy estimates

We do a little bit of tricks for the last one,

$$\begin{aligned}\left|\int v_2^\epsilon \partial_y \phi^{\epsilon,\alpha} \cdot v^\epsilon\right| &\le \left|\int v_2^\epsilon \partial_y \phi_1^{\epsilon,\alpha} v_1^\epsilon\right| + \left|\int v_2^\epsilon \partial_y \phi_2^{\epsilon,\alpha} v_2^\epsilon\right| \\ &\le \|\frac{v_2^\epsilon}{y}\|_{L^2}\|y^2\partial_y\phi_1^{\epsilon,\alpha}\|_{L^\infty}\|\frac{v_1^\epsilon}{y}\|_{L^2} + \|\partial_y\phi_2^{\epsilon,\alpha}\|_{L^\infty}\|v_2^\epsilon\|_{L^2}^2 \\ &\le C\|\partial_y v_2^\epsilon\|_{L^2}\epsilon^\alpha\|u^0\|_{L^\infty}\|\partial_y v_1^\epsilon\|_{L^2} + C\|u^0\|_{H^s}\|v_2^\epsilon\|_{L^2}^2 \\ &\le C\epsilon^\alpha\|\partial_x v^\epsilon\|_{L^2}\|u^0\|_{L^\infty}\|\partial_y v^\epsilon\|_{L^2} + C\|u^0\|_{H^s}\|v^\epsilon\|_{L^2}^2 \\ &\le \frac{\epsilon_1}{10}\|\partial_x v^\epsilon\|_{L^2}^2 + \frac{C\epsilon^{2\alpha}\|u^0\|_{L^\infty}^2}{\epsilon_1}\|\partial_y v^\epsilon\|_{L^2}^2 + C\|u\|_{H^s}\|v^\epsilon\|_{L^2}^2.\end{aligned}$$

The sketch of proof: energy estimates

Taking the L^2 norm of w^ϵ, we obtain that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w^\epsilon\|_{L^2}^2 + \epsilon_3\|\partial_x w^\epsilon\|_{L^2}^2 + \epsilon_4\|\partial_y w^\epsilon\|_{L^2}^2 \\
= &-\int \partial_t \psi^{\epsilon,\beta} w^\epsilon - \int v^\epsilon \cdot \nabla\theta^0 w^\epsilon - \int v^\epsilon \cdot \nabla\psi^{\epsilon,\beta} w^\epsilon - \int u^0 \cdot \nabla\psi^{\epsilon,\beta} w^\epsilon \\
&- \int \phi^{\epsilon,\alpha} \cdot \nabla\theta^0 w^\epsilon - \int \phi^{\epsilon,\alpha} \cdot \nabla\psi^{\epsilon,\beta} w^\epsilon + \epsilon_3 \int \partial_{xx}\theta^0 w^\epsilon \\
&+ \epsilon_3 \int \partial_{xx}\psi^{\epsilon,\beta} w^\epsilon + \epsilon_4 \int \partial_{yy}\theta^0 w^\epsilon + \epsilon_4 \int \partial_{yy}\psi^{\epsilon,\beta} w^\epsilon \\
&+ \epsilon_3 \int \partial_{xx}(a\rho(\frac{y}{\epsilon^\gamma}))w^\epsilon + \epsilon_4 \int \partial_{yy}(a\rho(\frac{y}{\epsilon^\gamma}))w^\epsilon - \int v^\epsilon \cdot \nabla(a\rho(\frac{y}{\epsilon^\gamma}))w^\epsilon \\
&- \int u \cdot \nabla(a\rho(\frac{y}{\epsilon^\gamma}))w^\epsilon - \int \phi^{\epsilon,\alpha} \cdot \nabla(a\rho(\frac{y}{\epsilon^\gamma}))w^\epsilon - \int a_t\rho(\frac{y}{\epsilon^\gamma})w^\epsilon.
\end{aligned}
$$

The sketch of proof: energy estimates

$$
\begin{aligned}
& -\int v^\epsilon \cdot \nabla \psi^{\epsilon,\beta} w^\epsilon \\
= & -\int v_1^\epsilon \partial_x \psi^{\epsilon,\beta} w^\epsilon - \int v_2^\epsilon \partial_y \psi^{\epsilon,\beta} w^\epsilon \\
\leq & \|v_1^\epsilon\|_{L^2} \|\partial_x \psi^{\epsilon,\beta}\|_{L^\infty} \|w^\epsilon\|_{L^2} + \|\frac{v_2^\epsilon}{y}\|_{L^2} \|y^2 \partial_y \psi^{\epsilon,\beta}\|_{L^\infty} \|\frac{w^\epsilon}{y}\|_{L^2} \\
\leq & C\|v_1^\epsilon\|_{L^2} \|\theta^0\|_{H^s} \|w^\epsilon\|_{L^2} + \|\partial_y v_2^\epsilon\|_{L^2} \|y^2 \partial_y \psi^{\epsilon,\beta}\|_{L^\infty} \|\partial_y w^\epsilon\|_{L^2} \\
\leq & C\|\theta^0\|_{H^s} \|v^\epsilon\|_{L^2} \|w^\epsilon\|_{L^2} + C\|\partial_x v^\epsilon\|_{L^2} \|\theta^0\|_{L^\infty} \epsilon^\beta \|\partial_y w^\epsilon\|_{L^2} \\
\leq & C\|\theta^0\|_{H^s} \|v^\epsilon\|_{L^2} \|w^\epsilon\|_{L^2} + \frac{\epsilon_1}{10} \|\partial_x v^\epsilon\|_{L^2}^2 + \frac{C\epsilon^{2\beta} \|\theta^0\|_{L^\infty}^2}{\epsilon_1} \|\partial_y w^\epsilon\|_{L^2}^2
\end{aligned}
$$

The sketch of proof: energy estimates

Taking all the bounds together, we have

$$\begin{aligned}
&\frac{d}{dt}(\|v^\epsilon\|_{L^2}^2 + \|w^\epsilon\|_{L^2}^2) + \epsilon_1\|\partial_x v^\epsilon\|_{L^2}^2 + \epsilon_2\|\partial_y v^\epsilon\|_{L^2}^2 + \epsilon_3\|\partial_x w^\epsilon\|_{L^2}^2 + \epsilon_4\|\partial_y w^\epsilon\|_{L^2}^2 \\
\leq& 2\eta_1\epsilon^{2\alpha}\|\partial_x v^\epsilon\|_{L^2}^2 + \frac{C\epsilon^{2\alpha}\|u^0\|_{L^\infty}^2}{\epsilon_1}\|\partial_y v^\epsilon\|_{L^2}^2 + C\frac{\epsilon^{2\beta}\|\theta^0\|_{L^\infty}^2 + \epsilon^{2\gamma}}{\epsilon_1}\|\partial_y w^\epsilon\|_{L^2}^2 \\
&+ C(\epsilon_3\epsilon^\gamma + \epsilon_4\epsilon^{-\gamma}) \\
&+ C(\epsilon_1 + \epsilon_1\epsilon^\alpha + \epsilon_2 + \epsilon_2\epsilon^{-\alpha})\|u^0\|_{H^s}^2 + C(\epsilon_3 + \epsilon_3\epsilon^\beta + \epsilon_4\epsilon^{-\beta} + \epsilon_4)\|\theta^0\|_{H^s}^2 \\
&+ C(\epsilon^{\frac{\alpha}{2}} + \epsilon^{\frac{\beta}{2}} + \epsilon^{\frac{\gamma}{2}} + \epsilon^{\alpha-\frac{\beta}{2}} + \epsilon^{\alpha-\frac{\gamma}{2}})(\|u^0\|_{H^s}^2 + \|\theta^0\|_{H^s}^2 + 1)(\|w^\epsilon\|_{L^2} + \|v^\epsilon\|_{L^2}) \\
&+ C(\|u^0\|_{H^s} + \|\theta^0\|_{H^s} + \frac{1}{\eta_1}\|u^0\|_{H^s}^2 + 1)(\|v^\epsilon\|_{L^2}^2 + \|w^\epsilon\|_{L^2}^2).
\end{aligned}$$

Assume

$$(\lambda_1 + \lambda_2)/2 < \alpha, \quad (\lambda_1 + \lambda_4)/2 < \gamma, \quad (\lambda_1 + \lambda_4)/2 < \beta,$$

and choose η_1 small, then the red in the right hand can be absorbed by the left.

The sketch of proof: energy estimates

It remains that

$$
\begin{aligned}
&\frac{d}{dt}(\|v^\epsilon\|_{L^2}^2+\|w^\epsilon\|_{L^2}^2)+\epsilon_1\|\partial_x v^\epsilon\|_{L^2}^2+\epsilon_2\|\partial_y v^\epsilon\|_{L^2}^2+\epsilon_3\|\partial_x w^\epsilon\|_{L^2}^2+\epsilon_4\|\partial_y w^\epsilon\|_{L^2}^2\\
\leq& C(\epsilon_1+\epsilon_2\epsilon^{-\alpha}+\epsilon_3+\epsilon_4\epsilon^{-\beta}+\epsilon_4\epsilon^{-\gamma})\\
&+C(\epsilon^{\frac{\alpha}{2}}+\epsilon^{\frac{\beta}{2}}+\epsilon^{\frac{\gamma}{2}}+\epsilon^{\alpha-\frac{\beta}{2}}+\epsilon^{\alpha-\frac{\gamma}{2}})(\|v^\epsilon\|_{L^2}+\|w^\epsilon\|_{L^2})+C(\|v^\epsilon\|_{L^2}^2+\|w^\epsilon\|_{L^2}^2).
\end{aligned}
$$

Applying Gronwall's lemma and and the estimates of $\phi^{\epsilon,\alpha}$ and $\psi^{\epsilon,\beta}$, we get the convergence. That is

$$\|u^\epsilon(t)-u(t)\|_{L^\infty(0,T_1;L^2)}\leq C(M)\tilde{\delta},$$

and

$$\|\theta^\epsilon(t)-\theta(t)\|_{L^\infty(0,T_1;L^2)}\leq C(M)\tilde{\delta},$$

where

$$\tilde{\delta}=\epsilon^{\frac{1}{2}\min\{\lambda_1,\ \lambda_2-\alpha,\ \lambda_3,\ \lambda_4-\beta,\ \lambda_4-\gamma,\ \beta,\ \gamma,\ 2\alpha-\beta,\ 2\alpha-\gamma\}}.$$

The sketch of proof: the convergence rate

Take

$$\beta = \gamma = (\frac{\lambda_1 + \lambda_4}{2})^+,$$

$$\alpha = \begin{cases} \max\{(\frac{\lambda_1+\lambda_2}{2})^+, \tilde{\alpha}\}, & \frac{\lambda_4+\lambda_1}{4} \leq \frac{\lambda_1+\lambda_2}{2}, \\ \tilde{\alpha}, & \frac{\lambda_4+\lambda_1}{4} > \frac{\lambda_1+\lambda_2}{2}, \end{cases}$$

where

$$\tilde{\alpha} = (\frac{\lambda_2}{3} + \frac{\lambda_4 + \lambda_1}{6})^+.$$

The exact convergence rate is

$$\delta = \epsilon^{\frac{1}{2}\min\{\lambda_1,\ \lambda_2-\alpha,\ \lambda_3,\ (\frac{\lambda_4-\lambda_1}{2})^-,\ (2\alpha-\frac{\lambda_4+\lambda_1}{2})^-\}}.$$

Introduction–anisotropic Boussinesq equations, Navier-slip and Neumann boundary conditions

Consider

$$
\begin{cases}
\partial_t u^\epsilon + u^\epsilon \cdot \nabla u^\epsilon + \nabla p^\epsilon = \epsilon_1 \partial_{xx} u^\epsilon + \epsilon_2 \partial_{yy} u^\epsilon + \theta^\epsilon e_2,\ t > 0,\ (x, y) \in \mathbb{R}^2_+ \\
\partial_t \theta^\epsilon + u^\epsilon \cdot \nabla \theta^\epsilon = \epsilon_3 \partial_{xx} \theta^\epsilon + \epsilon_4 \partial_{yy} \theta^\epsilon, \\
\nabla \cdot u^\epsilon = 0, \\
(D(u^\epsilon) n + a u^\epsilon) \cdot \tau\,|_{y=0} = 0, \quad u^\epsilon \cdot n\,|_{y=0} = 0, \\
\dfrac{\partial \theta^\epsilon}{\partial n}\,|_{y=0} = 0, \\
(u^\epsilon, \theta^\epsilon)|_{t=0} = (u_0, \theta_0),
\end{cases}
\tag{14}
$$

where $\epsilon_i = \epsilon^{\lambda_i},\ \lambda_i > 0,\ i = 1, \cdots, 4$, and $\epsilon \to 0$.

- How does ϵ_i affect the vanishing dissipation limit?
- Establishing $\|(u^\epsilon, \theta^\epsilon) - (u^0, \theta^0)\|_{L^\infty(0,T;L^2)} \to 0$, as $\epsilon \to 0$.
- Calculating the convergence rate.

Vanishing dissipation result

Theorem 4 (Qian Li, Peixin Wang, 2022)

Assume $(u_0, \theta_0) \in H^s(\mathbb{R}^2_+)$ *satisfies the corresponding compatibility conditions, with* $s > 3$, *integer. Let* $(u^\epsilon, \theta^\epsilon)$ *be the solution of* (14), (u^0, θ^0) *be the solution of* (9) (10) *and* $\gamma \leq \lambda_2/2$, *then*

$$\|(u^\epsilon, \theta^\epsilon) - (u^0, \theta^0)\|_{L^\infty(0,T_*;L^2(\mathbb{R}^2_+))} \leq C(T_*, u_0, \theta_0, a)\delta,$$

where the positive constant C *is independent of* ϵ, ϵ_i $(i = 1, \cdots, 4)$, *and the convergence rate* δ *is*

$$\delta = \epsilon^{\min\{\lambda_1,\ \frac{3\lambda_2}{4} - \gamma,\ \lambda_3,\ \frac{3\lambda_4}{4}\}}.$$

The sketch of proof: boundary layer correctors

Let $\tilde{\rho} \in C_0^\infty(\mathbb{R}_+)$, $\tilde{\rho}(0) = 1$, $\int_0^1 \tilde{\rho}(s)ds = \int_0^1 \int_0^\tau \tilde{\rho}(s)dsd\tau = 0$.
$\alpha, \beta > 0$ will be determined.

- Boundary layer corrector $\phi^{\epsilon,\alpha} = (\phi_1^{\epsilon,\alpha}, \phi_2^{\epsilon,\alpha})$ for velocity,

$$\phi_1^{\epsilon,\alpha}(t;x,y) := -\left[2a(t;x)u_1^0(t;x,0) + \partial_y u_1^0(t;x,0)\right] \int_0^y \tilde{\rho}\left(\frac{s}{\epsilon^\alpha}\right) ds,$$

$$\phi_2^{\epsilon,\alpha}(t;x,y) := \partial_x \left[2a(t;x)u_1^0(t;x,0) + \partial_y u_1^0(t;x,0)\right] \int_0^y \int_0^\tau \tilde{\rho}\left(\frac{s}{\epsilon^\alpha}\right) dsd\tau.$$

$\phi^{\epsilon,\alpha}$ satisfies

$$\begin{aligned} &\nabla \cdot \phi^{\epsilon,\alpha} = 0, \\ &\phi^{\epsilon,\alpha} \cdot n|_{y=0} = 0, \\ &\left[D(u^0 + \phi^{\epsilon,\alpha})n + a(u^0 + \phi^{\epsilon,\alpha})\right] \cdot \tau|_{y=0} = 0. \end{aligned}$$

The sketch of proof: boundary layer correctors

$$
\begin{aligned}
&\|\phi_1^{\epsilon,\alpha}\|_{L_y^2(\mathbb{R}_+)} \le C\epsilon^{3\alpha/2}, && \|\phi_2^{\epsilon,\alpha}\|_{L_y^2(\mathbb{R}_+)} \le C\epsilon^{5\alpha/2},\\
&\|\phi_1^{\epsilon,\alpha}\|_{L_y^\infty(\mathbb{R}_+)} \le C\epsilon^{\alpha}, && \|\phi_2^{\epsilon,\alpha}\|_{L_y^\infty(\mathbb{R}_+)} \le C\epsilon^{2\alpha},\\
&\|\partial_y\phi_1^{\epsilon,\alpha}\|_{L_y^2(\mathbb{R}_+)} \le C\epsilon^{\alpha/2}, && \|\partial_y\phi_1^{\epsilon,\alpha}\|_{L_y^\infty(\mathbb{R}_+)} \le C,\\
&\|y\partial_y\phi_1^{\epsilon,\alpha}\|_{L_y^2(\mathbb{R}_+)} \le C\epsilon^{3\alpha/2}, && \|y\partial_y\phi_1^{\epsilon,\alpha}\|_{L_y^\infty(\mathbb{R}_+)} \le C\epsilon^{\alpha}.
\end{aligned}
$$

- Boundary layer corrector for temperature

$$\psi^{\epsilon,\beta}(t;x,y) := -\partial_y\theta^0(t;x,0)\int_0^y \tilde{\rho}\left(\frac{s}{\epsilon^\beta}\right)ds.$$

$\psi^{\epsilon,\beta}$ satisfies

$$\left.\frac{\partial\left(\psi^{\epsilon,\beta}+\theta^0\right)}{\partial n}\right|_{y=0} = 0.$$

$$
\begin{aligned}
&\|\psi^{\epsilon,\beta}\|_{L_y^2(\mathbb{R}_+)} \le C\epsilon^{3\beta/2},\\
&\|\partial_y\psi^{\epsilon,\beta}\|_{L_y^2(\mathbb{R}_+)} \le C\epsilon^{\beta/2},\\
&\|\psi^{\epsilon,\beta}\|_{L_y^\infty(\mathbb{R}_+)} \le C\epsilon^{\beta}.
\end{aligned}
$$

The sketch of proof: energy estimates

Let $v^\epsilon = u^\epsilon - u^0 - \phi^{\epsilon,\alpha}$, $w^\epsilon = \theta^\epsilon - \theta^0 - \psi^{\epsilon,\beta}$, $q^\epsilon = p^\epsilon - p^0$.

$$\begin{aligned}&\partial_t v^\epsilon + \partial_t \phi^{\epsilon,\alpha} + u^0 \cdot \nabla(v^\epsilon + \phi^{\epsilon,\alpha}) + (v^\epsilon + \phi^{\epsilon,\alpha}) \cdot \nabla(v^\epsilon + u^0 + \phi^{\epsilon,\alpha}) + \nabla q^\epsilon \\ &= \epsilon_1 \partial_{xx}(v^\epsilon + u^0 + \phi^{\epsilon,\alpha}) + \epsilon_2 \partial_{yy}(v^\epsilon + u^0 + \phi^{\epsilon,\alpha}) + (\psi^{\epsilon,\beta} + w^\epsilon)\mathbf{e}_2,\end{aligned}$$

$$\begin{aligned}&\partial_t w^\epsilon + \partial_t \psi^{\epsilon,\beta} + (v^\epsilon + \phi^{\epsilon,\alpha}) \cdot \nabla(\theta^0 + \psi^{\epsilon,\beta} + w^\epsilon) + u^0 \cdot \nabla(\psi^{\epsilon,\beta} + w^\epsilon) \\ &= \epsilon_3 \partial_{xx}(\theta^0 + \psi^{\epsilon,\beta} + w^\epsilon) + \epsilon_4 \partial_{yy}(\theta^0 + \psi^{\epsilon,\beta} + w^\epsilon),\end{aligned}$$

$$\begin{gathered}\nabla \cdot v^\epsilon = 0, \\ v^\epsilon \cdot n|_{y=0} = 0, \quad [D(v^\epsilon)n + a v^\epsilon]_{\tan}|_{y=0} = 0, \\ \frac{\partial w^\epsilon}{\partial n}|_{y=0} = 0, \\ v^\epsilon|_{t=0} = -\phi^{\epsilon,\alpha}|_{t=0}, \quad w^\epsilon|_{t=0} = -\psi^{\epsilon,\beta}|_{t=0}.\end{gathered}$$

The sketch of proof: energy estimates

For L^2 energy estimates of v^ϵ and w^ϵ, it follows that

$$\frac{1}{2}\frac{d}{dt}||v^\epsilon||_{L^2}^2 + \epsilon_1||\partial_x v^\epsilon||_{L^2}^2 + \epsilon_2||\partial_y v^\epsilon||_{L^2}^2 = \cdots + \epsilon_2\int_{\mathbb{R}_+^2}\partial_{yy}\phi^{\epsilon,\alpha}\cdot v^\epsilon + \cdots$$

$$\begin{aligned}
&\epsilon_2\int_{\mathbb{R}_+^2}\partial_{yy}\phi^{\epsilon,\alpha}\cdot v^\epsilon \\
\leq&\left|\epsilon_2\int\partial_y\phi^{\epsilon,\alpha}\cdot\partial_y v^\epsilon\right| + \left|\epsilon_2\int_{\mathbb{R}}\partial_y\phi^{\epsilon,\alpha}\cdot v^\epsilon|_{y=0}dx\right| \\
\leq&\epsilon_2||\partial_y\phi^{\epsilon,\alpha}||_{L^2}||\partial_y v^\epsilon||_{L^2} + \left|\epsilon_2\int_{\mathbb{R}}-(2au_1^0+\partial_y u_1^0)|_{y=0}v_1^\epsilon|_{y=0}dx\right| \\
\leq&C\epsilon_2\epsilon^{\alpha/2}||u^0||_{H^3}||\partial_y v^\epsilon||_{L^2} + \epsilon_2||(2au_1^0+\partial_y u_1^0)|_{y=0}||_{L_x^2}||v_1^\epsilon|_{y=0}||_{L_x^2} \\
\leq&\frac{\epsilon_2}{8}||\partial_y v^\epsilon||_{L^2}^2 + C\epsilon_2\epsilon^\alpha||u^0||_{H^3}^2 + \epsilon_2||u^0||_{H^2}||v_1^\epsilon||_{L^2}^{1/2}||\partial_y v_1^\epsilon||_{L^2}^{1/2} \\
\leq&\frac{\epsilon_2}{4}||\partial_y v^\epsilon||_{L^2}^2 + C\epsilon_2\epsilon^\alpha||u^0||_{H^3}^2 + C\epsilon_2^{3/2}||u^0||_{H^2}^2 + ||v^\epsilon||_{L^2}^2.
\end{aligned}$$

The sketch of proof: energy estimates

$$\frac{1}{2}\frac{d}{dt}||w^\epsilon||^2_{L^2}+\epsilon_3||\partial_x w^\epsilon||^2_{L^2}+\epsilon_4||\partial_y w^\epsilon||^2_{L^2}=\cdots+\epsilon_4\int_{\mathbb{R}^2_+}\partial_{yy}\psi^{\epsilon,\beta}w^\epsilon+\cdots$$

$$\begin{aligned}
&\epsilon_4\int_{\mathbb{R}^2_+}\partial_{yy}\psi^{\epsilon,\beta}w^\epsilon\\
\leq&\left|\epsilon_4\int\partial_y\psi^{\epsilon,\beta}\partial_y w^\epsilon\right|+\left|\epsilon_4\int_{\mathbb{R}}\partial_y\psi^{\epsilon,\beta}w^\epsilon|_{y=0}dx\right|\\
\leq&\epsilon_4||\partial_y\psi^{\epsilon,\beta}||_{L^2}||\partial_y w^\epsilon||_{L^2}+\left|\epsilon_4\int_{\mathbb{R}}\partial_y\theta^0|_{y=0}w^\epsilon|_{y=0}dx\right|\\
\leq&\epsilon_4\epsilon^{\beta/2}||\theta^0||_{H^2}||\partial_y w^\epsilon||_{L^2}+\epsilon_4||\partial_y\theta^0|_{y=0}||_{L^2_x}||w^\epsilon|_{y=0}||_{L^2_x}\\
\leq&\epsilon_4\epsilon^{\beta/2}||\theta^0||_{H^2}||\partial_y w^\epsilon||_{L^2}+\epsilon_4||\theta^0||_{H^2}||w^\epsilon||^{1/2}_{L^2}||\partial_y w^\epsilon||^{1/2}_{L^2}\\
\leq&\frac{\epsilon_4}{2}||\partial_y w^\epsilon||^2_{L^2}+C\epsilon_4\epsilon^\beta||\theta^0||^2_{H^2}+C\epsilon_4^{3/2}||\theta^0||^2_{H^2}+||w^\epsilon||^2_{L^2}.
\end{aligned}$$

The sketch of proof: energy estimates

$$
\begin{aligned}
&\frac{d}{dt}(\|v^\epsilon(t)\|_{L^2}^2+\|w^\epsilon(t)\|_{L^2}^2)\\
&+\epsilon_1\|\partial_x v^\epsilon\|_{L^2}^2+\epsilon_2\|\partial_y v^\epsilon\|_{L^2}^2+\epsilon_3\|\partial_x w^\epsilon\|_{L^2}^2+\epsilon_4\|\partial_y w^\epsilon\|_{L^2}^2\\
\le& C\left(1+\epsilon^\alpha+\epsilon^\beta+\cancel{\frac{\epsilon^{4\alpha}}{\epsilon_1}}+\epsilon_2\right)(\|v^\epsilon\|_{L^2}^2+\|w^\epsilon\|_{L^2}^2)\\
&+C(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\cancel{\epsilon^{3\alpha/2}}+\cancel{\epsilon^{3\beta/2}})(\|v^\epsilon\|_{L^2}+\|w^\epsilon\|_{L^2})\\
&+C(\epsilon_1\epsilon^{3\alpha}+\epsilon_2\epsilon^\alpha+\epsilon_2^{3/2}+\epsilon_3\epsilon^{3\beta}+\epsilon_4\epsilon^\beta+\epsilon_4^{3/2}).
\end{aligned}
$$

Applying Gronwall's lemma and and the estimates of $\phi^{\epsilon,\alpha}$ and $\psi^{\epsilon,\beta}$, we get

$$
\|u^\epsilon-u^0\|_{L^\infty(0,T_*;L^2)}+\|\theta^\epsilon-\theta^0\|_{L^\infty(0,T_*;L^2)}\le C\delta,
$$

$$
\begin{aligned}
\delta=&\max\left\{\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4,(\epsilon_1\epsilon^{3\alpha}+\epsilon_2\epsilon^\alpha+\epsilon_2^{3/2}+\epsilon_3\epsilon^{3\beta}+\epsilon_4\epsilon^\beta+\epsilon_4^{3/2})^{1/2}\right\}\\
=&\epsilon^{\min\left\{\lambda_1,\frac{3\lambda_2}{4},\lambda_3,\frac{3\lambda_4}{4}\right\}}.
\end{aligned}
$$

Physical phenomenon

Dirichlet boundary conditions:

- The vertical dissipation (ϵ_2, ϵ_4) play a more dominant role in the vanishing dissipation limit.
- ϵ_2, ϵ_4 more rapidly tending to zero will cause the vanishing of the boundary layer in L^2 norm.

Navier-slip and Neumann boundary conditions:

- The anisotropic dissipation coefficients have almost equal effects on the vanishing dissipation limit.
- Boundary layer must vanish in L^2 norm.

Introduction–Boussinesq equations, Dirichlet boundary conditions

$$
\begin{cases}
\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon = \nu \Delta u^\varepsilon + \theta^\varepsilon \mathbf{e}_2,\ t > 0,\ (x, y) \in \mathbb{R}^2_+, \\
\partial_t \theta^\varepsilon + u^\varepsilon \cdot \nabla \theta^\varepsilon = \varepsilon \Delta \theta^\varepsilon, \\
\nabla \cdot u^\varepsilon = 0, \\
u^\varepsilon|_{y=0} = 0, \quad \theta^\varepsilon|_{y=0} = a(t; x), \\
(u^\varepsilon, \theta^\varepsilon)|_{t=0} = (u_0, \theta_0).
\end{cases}
\tag{15}
$$

- Temperature boundary layer exists.
  (Jiang–Zhang–Zhao, J. Differ. Equ., 2011)
- Question:
  Does the temperature boundary layer cause the velocity boundary layer?

  The boundary layers are weak or strong?

  The asymptotic behavior of $(u^\epsilon, \theta^\epsilon)$ in $L^\infty((0, T) \times \mathbb{R}^2_+)$, as $\varepsilon \to 0$?

Global well-posedness

Let $\varepsilon = 0$, (15) formally becomes

$$
\begin{cases}
\partial_t u^{I,0} + u^{I,0} \cdot \nabla u^{I,0} + \nabla p^{I,0} = \nu \Delta u^{I,0} + \theta^{I,0} \mathbf{e}_2, \\
\partial_t \theta^{I,0} + u^{I,0} \cdot \nabla \theta^{I,0} = 0, \\
\nabla \cdot u^{I,0} = 0, \\
u^{I,0}|_{y=0} = 0, \\
(u^{I,0}, \theta^{I,0})|_{t=0} = (u_0, \theta_0).
\end{cases}
\tag{16}
$$

Theorem 5 (Wang–Xu, 2022)

Assume $(u_0, \theta_0) \in H^3(\mathbb{R}^2_+)$, *and* (u_0, θ_0) *satisfies the corresponding compatibility conditions. Then* (16) *admits a unique global strong solution* $(u^{I,0}, \theta^{I,0})$, *such that* $\forall T > 0$,

$$
u^{I,0} \in L^\infty(0, T; H^3(\mathbb{R}^2_+)) \cap L^2(0, T; H^4(\mathbb{R}^2_+)),
$$
$$
\theta^{I,0} \in L^\infty(0, T; H^3(\mathbb{R}^2_+)).
$$

Vanishing diffusivity limit ($\theta^\varepsilon|_{y=0} = a(t;x)$)

$$\theta^{I,0}(t;x,0) = \theta^{I,0}(0;x,0) = \theta_0(x,0) = a(0;x)$$

$$\Longrightarrow \theta^\varepsilon|_{y=0} \neq \theta^{I,0}|_{y=0}.$$

Theorem 6 (Wang–Xu, 2022)

Assume $(u_0,\theta_0) \in H^s(\mathbb{R}^2_+)$, $a(t;x) \in C^{[\frac{s+1}{2}]+1}([0,+\infty), H^s(\mathbb{R}))$ *for* $s \geq 23$, *and* (u_0,θ_0) *satisfies the corresponding compatibility conditions. Let* $(u^\varepsilon,\theta^\varepsilon)$ *be the solution to* (15), *and* $(u^{I,0},\theta^{I,0})$ *be the solution to* (16). *Then, there exists a boundary layer profile* $\theta^{B,0}$, *such that for any* $T > 0$, *the expansions*

$$u^\varepsilon(t;x,y) = u^{I,0}(t;x,y) + \gamma^\varepsilon(t;x,y),$$
$$\theta^\varepsilon(t;x,y) = \theta^{I,0}(t;x,y) + \theta^{B,0}(t;x,\frac{y}{\sqrt{\varepsilon}}) + \delta^\varepsilon(t;x,y)$$

hold in $L^\infty((0,T)\times\mathbb{R}^2_+)$, *with the remainder* $\gamma^\varepsilon = O(\varepsilon^{\frac{3}{4}})$, $\delta^\varepsilon = O(\varepsilon^{\frac{1}{4}})$.

The sketch of proof. Step1: asymptotic expansions

Take the following ansatz for $(u^\varepsilon, p^\varepsilon, \theta^\varepsilon)$:

$$\begin{cases} u^\varepsilon(t;x,y) = \sum_{j\geq 0} \varepsilon^{\frac{j}{2}} \left(u^{I,j}(t;x,y) + u^{B,j}(t;x,\eta)\right), \\ p^\varepsilon(t;x,y) = \sum_{j\geq 0} \varepsilon^{\frac{j}{2}} \left(p^{I,j}(t;x,y) + p^{B,j}(t;x,\eta)\right), \\ \theta^\varepsilon(t;x,y) = \sum_{j\geq 0} \varepsilon^{\frac{j}{2}} \left(\theta^{I,j}(t;x,y) + \theta^{B,j}(t;x,\eta)\right), \end{cases}$$

where $\eta = \frac{y}{\sqrt{\varepsilon}}$, and as $\eta \to +\infty$, $(u^{B,j}, p^{B,j}, \theta^{B,j})(t;x,\eta)$ decays to zero exponentially.

$$\begin{aligned} O(\varepsilon^{-1}): &\quad u^{B,0} = 0, \quad u_2^{B,1} = 0. \\ O(\varepsilon^{-\frac{1}{2}}): &\quad u_1^{B,1} = 0, \quad u_2^{B,2} = 0, \quad p^{B,0} = 0. \\ O(\varepsilon^{0}): &\quad u_1^{B,2} = 0, \quad u_2^{B,3} = 0, \quad p^{B,1} \neq 0, \quad \theta^{B,0} \neq 0. \\ O(\varepsilon^{\frac{1}{2}}): &\quad u_1^{B,3} \neq 0, \quad u_2^{B,4} \neq 0, \quad \cdots \end{aligned}$$

Strong boundary layer profiles: $u^{B,0}$, $\theta^{B,0}$.

Step2: construct approximate solution to (15)

Let

$$\begin{cases} u^{\varepsilon,\alpha}(t;x,y) = \sum_{j=0}^{5} \varepsilon^{\frac{j}{2}} u^{I,j}(t;x,y) + \sum_{j=0}^{4} \varepsilon^{\frac{j}{2}} u^{B,j}(t;x,\frac{y}{\sqrt{\varepsilon}}) \\ \qquad + \varepsilon^{\frac{5}{2}} \left(\begin{pmatrix} 0 \\ u_2^{B,5}(t;x,\frac{y}{\sqrt{\varepsilon}}) \end{pmatrix} + \begin{pmatrix} -u_1^{I,5}|_{y=0}\phi(\frac{y}{\sqrt{\varepsilon}}) \\ \sqrt{\varepsilon}\partial_x u_1^{I,5}|_{y=0}\psi(\frac{y}{\sqrt{\varepsilon}}) \end{pmatrix} \right), \\ p^{\varepsilon,\alpha}(t;x,y) = \sum_{j=0}^{4} \varepsilon^{\frac{j}{2}} p^{I,j}(t;x,y) + \sum_{j=0}^{4} \varepsilon^{\frac{j}{2}} p^{B,j}(t;x,\frac{y}{\sqrt{\varepsilon}}), \\ \theta^{\varepsilon,\alpha}(t;x,y) = \sum_{j=0}^{4} \varepsilon^{\frac{j}{2}} \theta^{I,j}(t;x,y) + \sum_{j=0}^{3} \varepsilon^{\frac{j}{2}} \theta^{B,j}(t;x,\frac{y}{\sqrt{\varepsilon}}) - \varepsilon^{\frac{4}{2}} \theta^{I,4}|_{y=0}\phi(\frac{y}{\sqrt{\varepsilon}}). \end{cases}$$

$\phi, \psi \in C_c^\infty([0,\infty))$, $\operatorname{supp}\phi, \operatorname{supp}\psi \subset [0,1]$, $\phi(0) = 1$, $\psi(0) = 0$ and $\psi'(y) = \phi(y)$.

Step3: energy estimates for errors

Let

$$\begin{cases} u^\varepsilon(t;x,y) = u^{\varepsilon,\alpha}(t;x,y) + \varepsilon^{4/2} R^\varepsilon(t;x,y), \\ p^\varepsilon(t;x,y) = p^{\varepsilon,\alpha}(t;x,y) + \varepsilon^{4/2} \pi^\varepsilon(t;x,y), \\ \theta^\varepsilon(t;x,y) = \theta^{\varepsilon,\alpha}(t;x,y) + \varepsilon^{4/2} w^\varepsilon(t;x,y), \end{cases}$$

then $(R^\varepsilon, \pi^\varepsilon, w^\varepsilon)$ satisfies

$$\begin{cases} \partial_t R^\varepsilon + u^\varepsilon \cdot \nabla R^\varepsilon + R^\varepsilon \cdot \nabla u^{\varepsilon,\alpha} + \nabla \pi^\varepsilon - \Delta R^\varepsilon - w^\varepsilon \mathbf{e}_2 = F_1^\varepsilon, \\ \partial_t w^\varepsilon + u^\varepsilon \cdot \nabla w^\varepsilon + R^\varepsilon \cdot \nabla \theta^{\varepsilon,\alpha} - \varepsilon \Delta w^\varepsilon = F_2^\varepsilon, \\ \nabla \cdot R^\varepsilon = 0, \\ R^\varepsilon(0;x,y) = 0, \quad w^\varepsilon(0;x,y) = 0, \\ R^\varepsilon(t;x,0) = 0, \quad w^\varepsilon(t;x,0) = 0, \end{cases} \tag{17}$$

where $F_1^\varepsilon = O(\varepsilon^{-\frac{1}{4}})$, $F_2^\varepsilon = O(\varepsilon^{\frac{1}{4}})$.

- $\|R^\varepsilon\|_{L^\infty((0,T)\times\mathbb{R}_+^2)} \leq C\varepsilon^{-\frac{5}{4}}, \quad \|w^\varepsilon\|_{L^\infty((0,T)\times\mathbb{R}_+^2)} \leq C\varepsilon^{-\frac{7}{4}}.$
  $\Longrightarrow u^\varepsilon = u^{\varepsilon,\alpha} + O(\varepsilon^{\frac{3}{4}}), \theta^\varepsilon = \theta^{\varepsilon,\alpha} + O(\varepsilon^{\frac{1}{4}})$ hold in $L^\infty_{t,x}$.

Vanishing diffusivity limit ($\theta^\varepsilon|_{y=0} = a(x)$)

$$\theta^{I,0}(t;x,0) = \theta^{I,0}(0;x,0) = \theta_0(x,0) = a(x)$$

$$\Longrightarrow \theta^\varepsilon|_{y=0} = \theta^{I,0}|_{y=0}.$$

Theorem 7 (Wang–Xu, 2022)

Assume $(u_0, \theta_0) \in H^s(\mathbb{R}^2_+)$, $a(x) \in H^s(\mathbb{R})$ *for* $s \geq 11$, *and* (u_0, θ_0) *satisfies the corresponding compatibility conditions. Let* $(u^\varepsilon, \theta^\varepsilon)$ *be the solution to* (15), *and* $(u^{I,0}, \theta^{I,0})$ *be the solution to* (16). *Then, for any* $T > 0$, *the expansions*

$$u^\varepsilon(t;x,y) = u^{I,0}(t;x,y) + r^\varepsilon(t;x,y),$$
$$\theta^\varepsilon(t;x,y) = \theta^{I,0}(t;x,y) + s^\varepsilon(t;x,y)$$

hold in $L^\infty((0,T) \times \mathbb{R}^2_+)$, *with the remainder* $r^\varepsilon = O(\varepsilon^{\frac{3}{4}})$, $s^\varepsilon = O(\varepsilon^{\frac{1}{4}})$.

Asymptotic expansions:

$$u_1^{B,i} = u_2^{B,j} = \theta^{B,l} = p^{B,k} = 0,\ 0 \leq i \leq 4,\ 0 \leq j \leq 5,\ 0 \leq l \leq 1,\ 0 \leq k \leq 2.$$

$\theta^{B,2} \neq 0$, $u^{B,3} \neq 0$, $u^{B,5} \neq 0$, $u^{B,6} \neq 0$

Physical phenomenon

- Strong temperature boundary layer occurs when temperature is changing with time on the boundary.
- Weak temperature boundary layer appears when temperature does not depend on time on the boundary.
- Only weak velocity boundary layers appear.

Thanks!