5.2 Newton strata

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$g \geq 1$ $\quad \mathrm{Newt}(g) := \left\{ \lambda_1 \leq \cdots \leq \lambda_{2g}, \quad \begin{array}{l} \lambda_i \in \mathbb{Q} \cap [0,1] \\ \lambda_i + \lambda_{2g+1-i} = 1 \\ \text{\& integral condition} \end{array} \right\}$

A: abelian variety $/ k = \bar{k}$ of dim g

$\because A \sim A^t \Rightarrow \mathrm{slope}(A) \in \mathrm{Newt}(g)$.

Def (1) For each $b \in \mathrm{Newt}(g)$, define

$$N_b := \{ (A, \lambda) \in \mathcal{A}_g(k) : \mathrm{slope}(A) = b \}.$$

the Newton stratum ass to b.

Similarly, we define the Newton stratum $N_b \subseteq \mathcal{A}_{g,N}$.

(2) $b = (\lambda_i)$, $b' = (\lambda_i') \in \mathrm{Newt}(g)$, define $b \preceq b'$ if the Newton polygon of b lies above or equals to that of b'.

Equivalently, if $\forall 1 \leq k \leq 2g-1$,

$$\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \lambda_i', \quad \text{and} \quad \sum_{i=1}^{2g} \lambda_i = \sum_{i=1}^{2g} \lambda_i' = g$$

Thm (1) $\forall\, b \in \mathrm{Newt}(g)$, $\mathcal{N}_b$ is non-empty, a locally closed subset of $\mathcal{A}_g$, regarded as an alg subvariety, or reduced subscheme of $\mathcal{A}_g$.

(2) The ordinary locus $\mathcal{A}_g^{\mathrm{ord}} \subset \mathcal{A}_g$ is Zariski (open) dense.

(3) The closure $\overline{\mathcal{N}_b} = \bigsqcup_{b' \preceq b} \mathcal{N}_{b'}$. (stratification property).

(4) If $b' \precneqq b$ is saturated, i.e $\nexists\, b''$ $b' \precneqq b'' \precneqq b$

then $\dim \mathcal{N}_{b'} = \dim \mathcal{N}_b - 1$.

(5) If $\mathcal{N}_b$ is not the supersingular locus, then $\mathcal{N}_b$ is irreducible.

(6) Denote by S_g the supersingular locus and $\Sigma(S_g)$ the set of geometrically irreducible components. Then

$$\dim S_g = [g^2/4], \quad \text{and}$$

$$\#\Sigma(S_g) = \begin{cases} H_g(p,1), & \text{if } g: \text{odd}; \\ H_g(1,p), & \text{if } g: \text{even}. \end{cases}$$

where $H_g(p,1)$ = the class # of the principal genus of hermitian O_B-lattices of rank g.

$H_g(1,p)$ = the class # of the non-principal genus of hermitian O_B-lattices of rank g.

$B = B_{p,\infty}$ the definite quaternion algebra $/\mathbb{Q}$, ramified at $\{p, \infty\}$

$\cup$

O_B : a maximal order. $\simeq \mathrm{End}(E)$, E: supersingular elliptic curve.

(7) There is a foliation structure on each Newton stratum where the leaves are charterized by the isom. classes of ass. polarized p-divisible groups.

- For the ordinary case : the whole Newton stratum is a leaf.
- For the supersingular case, every leaf consists of finitely many points.

Remark (1) Non-emptiness of Newton strata is known as Manin's problem. It is proved by Tate using the Honda-Tate Theorem.

(2) This is first proved by Koblitz in his thesis (1975), where he shows more results on p-rank strata:

For $0 \leq f \leq g$, $\mathcal{A}_g^{(f)} := \{(A, \lambda) \in \mathcal{A}_g(k) : p\text{-rank}(A) = f\}$

Note $\mathcal{A}_g^{(f)}$ = union of some Newton strata.

Koblitz showed ① $\dim \mathcal{A}_g^{(f)} = g(g-1)/2 + f$.

② $\mathcal{A}_g^{(f)} \subseteq \mathcal{A}_g^{\leq f}$ Zariski dense.

Later Norman-Oort (80, Ann. Math) generalized Koblitz's results to moduli spaces of arbitrary polarized. abelian varieties. In particular.

$\mathcal{A}^{ord}_{g,d} \subseteq \mathcal{A}_{g,d}$ is Zariski dense. This is how one shows that

every polarized abelian variety in char p so can be lifted to char 0.

(3) Grothendieck's specialization thm: if $\mathcal{N}_{b'} \cap \overline{\mathcal{N}_b} \neq \phi$, then $b' \preceq b$

The converse : If $b' \preceq b$, then $\mathcal{N}_{b'} \subseteq \overline{\mathcal{N}_b}$ (Oort's Thm).

The Grothendieck conjecture for Newton strata also holds for a class of good reduction of more general Shimura varieties. However, it does not hold for bad reduction of some Shimura varieties. e.g. Stamm's thesis (1993).

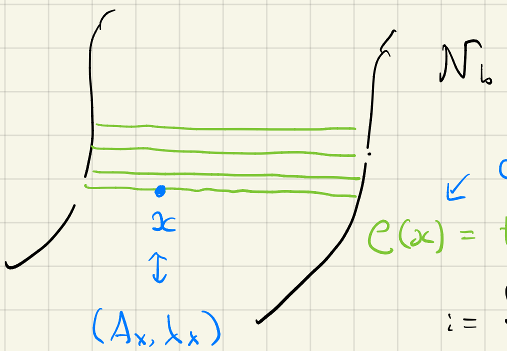
(4) This is predicted from the purity of Newton strata (de Jong & Oort)

(5) This is due to Chai-Oort (2011), which is a consequence of the discrete part of HO conjecture (proved by them).

(6) For $g=2,3$, this is due to Ibukiyama-Katsura-Oort, and Katsura-Oort. For general g, K.Z. Li and Oort. Note that computation of $H_g(p,1)$ or $H_g(1,p)$ is very difficult. Known for $g=2$ (Hashimoto-Ibukiyama) $H_3(p,1)$: (Hashimoto), $S_g(a) = \{(A,\lambda) \in S_g,\ a(A) = a\}$. Then $S_g(1) \subseteq S_g$ Zariski dense. Similar result also holds for other Newton strata :

If $b \neq \mathrm{ord}$, then $\mathcal{N}_b(1) \subset \mathcal{N}_b$ Zariski dense. This is an important step in Oort's proof of Grothendieck's conjecture.

(7)



central leaf

$C(x)$ = the leaf passing x

$$:= \{(A, \lambda) \in \mathcal{A}_g : (A,\lambda)[p^\infty] \simeq (A_x,\lambda_x)[p^\infty]\}$$

More precisely. $\forall b$. $\forall$ irreducible component $W \subseteq \mathcal{N}_b$, $\exists$ irreducible algebraic varieties T and J and a finite surjective morphism

$$\Phi : T \times J \twoheadrightarrow W \quad \text{s.t.}$$

$\forall u \in J(k)$, $\Phi(T \times \{u\})$ is a central leaf in W and every central leaf in W can be obtained in this way.

$\forall t \in T(k)$, $\Phi(\{t\} \times J)$ is an "isogeny leaf" in W, and every "isogeny leaf" in W can be obtained in this way.

§6 Truncated Barsotti-Tate groups and Dieudonné modules.

6.1 Mod p of polarized Dieudonné modules.

Def: (1) A polarized Dieudonné module is a pair $(M, \langle , \rangle)$, where

M is a W-free Dieudonné module and

$\langle , \rangle : M \times M \to W$: non-degenerate W-bilinear alternating pairing

s.t. $\langle Fx, y \rangle = \langle x, Vy \rangle^{\sigma}$ $\forall\ x, y \in M$.

(equivalently, $\langle Vx, y \rangle = \langle x, Fy \rangle^{\sigma^{-1}}$ or $\langle Fx, Fy \rangle = p \langle x, y \rangle^{\sigma}$)

(2) If $\langle , \rangle$ is perfect, i.e. $\langle , \cdot \rangle : M \xrightarrow{\sim} M^t$ is an isom, then

$(M, \langle , \rangle)$ is called a principally polarized Dieudonné module.

(3) If $\langle , \rangle : M \times M \to B(k)$ takes values in $B(k)$ instead of W,

then $(M, \langle , \rangle)$ is called a $\mathbb{Q}$-polarized Dieudonné module.

Note: In the literature $\langle , \rangle$ in (1) is also called a quasi-polarization

and $(M, \langle , \rangle)$ in (2) is called a separably quasi-polarized DM.

(4) A truncated Dieudonné module of level n or called a DM_n

is a Dieudonné module of finite length which is $M/p^n M$ for some

(W-free) Dieudonné module M. Similarly, a truncated Barsotti-Tate

group or a BT_n is a finite group scheme which is $G[p^n]$

for a p-divisible group G.

(5) A principally polarized DM_n is of form $(M/p^nM, \langle,\rangle_n)$ for a p.p. DM $(M, \langle,\rangle)$. Similarly, a p.p. BT_n is of the form $(G[p^n], \lambda_n)$ for a p.p. p-divisible group (G, λ).

Note: a polarization on G is an isogeny $\lambda: G \to G^t$ satisfying $\lambda^t = -\lambda$.

Prop. (1) A DM_1 over k is a fin-dim'l k-vect. space $\bar{M}$ together a σ-linear map F and a σ^{-1}-linear map V s.t.

$$\ker F = \operatorname{Im} V \quad \text{and} \quad \ker V = \operatorname{Im} F.$$

(2) A pp DM_1 over is a symplectic space over k $(\bar{M}, \langle,\rangle)$, where $\bar{M}$ is a DM_1 s.t. $\langle Fx, y\rangle = \langle x, Vy\rangle^p$ & $\langle Vx, y\rangle = \langle x, Fy\rangle^{p^{-1}}$, $x, y \in \bar{M}$.

6.2 Examples:

(1) $M = W\{e_1, e_2\}$ $\quad F: \begin{array}{l} e_1 \mapsto e_1 \\ e_2 \mapsto pe_2 \end{array}$ $\quad V: \begin{array}{l} e_1 \to pe_1 \\ e_2 \to e_2 \end{array}$

$\langle e_1, e_2\rangle = 1 = -\langle e_2, e_1\rangle$, other pairings $= 0$.

$\boxed{\langle Fe_i, e_j\rangle = \langle e_i, Ve_j\rangle^\sigma \quad \forall i,j.}$ ok : $i=1, j=1$ LHS = RHS = 0

$\langle Fe_1, e_2\rangle = \langle e_1, e_2\rangle = 1$, $\quad \langle e_1, Ve_2\rangle^\sigma = \langle e_1, e_2\rangle^\sigma = 1$.

(2) $M = W\{e_1, e_2\}$, $\quad F: \begin{array}{l} e_1 \mapsto e_2 \\ e_2 \mapsto -pe_1 \end{array}$ $\quad V: \begin{array}{l} e_1 \mapsto -e_2 \\ e_2 \mapsto pe_1 \end{array}$

$\langle e_1, e_2\rangle = 1 = -\langle e_2, e_1\rangle$

$i=1, j=2$. $\quad 0 = 0$

$\langle Fe_i, e_j\rangle = \langle e_i, Ve_j\rangle^\sigma$

$\langle Fe_1, e_1\rangle = \langle e_2, e_1\rangle = -1 = \langle e_1, -e_2\rangle = \langle e_1, Ve_1\rangle$

ok.

(3) General construction: M: free W-module of rank $2g$.

choose two W-bases $x_1, \cdots x_{2g}$

$y_1 \cdots y_{2g}$.

define

$$F: \begin{array}{l} x_1 \mapsto y_1 \\ \vdots \\ x_g \mapsto y_g \\ x_{g+1} \mapsto p\cdot y_{g+1} \\ \vdots \\ x_{2g} \mapsto p\cdot y_{2g}. \end{array} \qquad V: \begin{array}{l} y_1 \mapsto p\cdot x_1 \\ \vdots \\ y_g \mapsto p\cdot x_g \\ y_{g+1} \mapsto x_{g+1} \\ \vdots \\ y_{2g} \mapsto x_{2g}. \end{array}$$

$\langle x_i, x_{g+i} \rangle = 1 = - \langle x_{g+i}, x_i \rangle \quad \forall\ 1 \leq i \leq g$, others $= 0$.

$$[y_1, \cdots, y_{2g}] = [x_1, \cdots x_{2g}] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in GL_{2n}(W),$$

$$F\cdot[x_1, \cdots, x_n] = [y_1, \cdots, y_g, py_{g+1}, \cdots, py_{2g}]$$

$$= [x_1, \cdots x_{2g}] \begin{bmatrix} A & B \\ C & D \end{bmatrix} \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & p \end{array}\right]$$

$$= [x_1, \cdots, x_{2g}] \begin{bmatrix} A & pB \\ C & pD \end{bmatrix}.$$

With W-basis $x_1, \cdots, x_{2g}$. $F \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 & \\ & p \end{bmatrix} \sigma = P\sigma$

$M = W^{2g}$: column vectors.

$$(\langle Fx_i\ Fx_j \rangle) = (p\langle x_i, x_j \rangle^\sigma) = p \underbrace{\left[\begin{array}{c|c} & I_g \\ \hline -I_g & \end{array}\right]}_{J_g}$$

$$\langle (a_i), (b_i) \rangle = (a_i)^t \cdot J_g\ (b_i).$$

$$\langle P(\sigma(a_i)), P(\sigma(b_i)) \rangle = (\sigma(a_i))^t \cdot \underbrace{P^t \cdot J_g P}(\sigma(b_i)), \quad P_0 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$= (\sigma(a_i))^t \cdot \underbrace{(P \cdot J_g)}(\sigma(b_i))$$

$$\Longleftrightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix}^t \cdot J_g \begin{bmatrix} A & B \\ C & D \end{bmatrix} = J_g \Rightarrow \boxed{\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp_{2g}(W).}$$

Suppose $X_1', \cdots, X_{2g}'$ another Lagrangian W- basis satisfying

$$X_{g+1}', \cdots, X_{2g}' \in VM$$

$$[X_1', \cdots, X_{2g}'] = [X_1, \cdots, X_{2g}] \cdot P, \quad P = \begin{bmatrix} A_1 & pB_1 \\ C_1 & D_1 \end{bmatrix} \in Sp_{2g}(W)$$

$$\downarrow$$

$$F[X_1', \cdots, X_{2g}'] = F[X_1, \cdots, X_{2g}] \cdot \sigma(P) \qquad P^* = \begin{bmatrix} A_1 & B_1 \\ pC_1 & D_1 \end{bmatrix}$$

$$= [X_1, \cdots, X_{2g}] \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 & \\ & p \end{bmatrix} \sigma(P)$$

$$= [X_1', \cdots X_{2g}'] \cdot P^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \\ & p \end{bmatrix} \sigma(p) \begin{bmatrix} 1 & \\ & p^{-1} \end{bmatrix}}_{\sigma(P)^*} \begin{bmatrix} 1 & \\ & p \end{bmatrix}$$

$$\Rightarrow F \cdot [X_1', \cdots, X_{2g}'] = [X_1', \cdots, X_{2g}'] P^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \sigma(P)^* \cdot \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}$$

$$= [X_1', \cdots, X_{2g}'] \cdot \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}$$

Prop. $(M, \langle , \rangle)$, $(M', \langle , \rangle')$. 2 p.p. DMs of rank $2g$.

Suppose $(M, \langle , \rangle)$ and $(M', \langle , \rangle')$ are defined by the matrices

$$P_0 = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ and } P_0' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \text{ in } Sp_{2g}(W), \text{ resp.}$$

(1) $(M, \langle , \rangle) \simeq (M', \langle , \rangle') \iff \exists\, P = \begin{bmatrix} A_1 & pB_1 \\ C_1 & D_1 \end{bmatrix} \in Sp_{2g}(W)$

s.t. $P_0' = P^{-1} \cdot P_0\, \sigma(P)^*$.

(2) $(\bar{M}, \langle , \rangle) \simeq (\bar{M}', \langle , \rangle')$ as pp DM,

$$\iff \exists\, Q_1 = \begin{bmatrix} \overbrace{\bar{A}_1}^{g} & \bar{B}_1 \\ 0 & \bar{D}_1 \end{bmatrix},\ Q_2 = \begin{bmatrix} \overbrace{\bar{A}_2}^{g} & 0 \\ \bar{C}_2 & \bar{D}_2 \end{bmatrix} \text{ in } Sp_{2g}(k) \text{ with}$$

$\bar{A}_1 \cdot \bar{A}_2 = id = \bar{D}_1 \cdot \bar{D}_2$ s.t. $\bar{P}_0' = Q_2 \cdot \bar{P}_0 \cdot \sigma(Q_1)$

Pf (1) follows from our computation

(2) Write $P = \begin{bmatrix} A_1 & pB_1 \\ C_1 & D_1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} A_2 & pB_2 \\ C_2 & D_2 \end{bmatrix}$

$$\begin{bmatrix} A_1 & pB_1 \\ C_1 & D_1 \end{bmatrix}\begin{bmatrix} A_2 & pB_2 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} A_1A_2 + pB_1C_2 & p(A_1B_2 + B_1D_2) \\ C_1A_2 + D_1C_2 & D_1D_2 + pC_1B_1 \end{bmatrix} = \begin{bmatrix} id & 0 \\ 0 & id \end{bmatrix}$$

Note $(\bar{M}, \langle , \rangle) \simeq (\bar{M}', \langle , \rangle') \iff \exists$ lifting $(M_1, \langle , \rangle_1)$ of $(\bar{M}, \langle , \rangle)$

& $(M_2, \langle , \rangle_2)$ of $(\bar{M}', \langle , \rangle')$

s.t. $(M_1, \langle , \rangle_1) \simeq (M_2, \langle , \rangle_2)$.

$\Rightarrow \quad \bar{A}_1 \cdot \bar{A}_2 = \mathrm{id}, \quad \bar{D}_1 \cdot \bar{D}_2 = \mathrm{id}.$

$$P^{-1} \square \sigma(P)^*$$

$$\begin{pmatrix} A_2 & pB_2 \\ C_2 & D_2 \end{pmatrix} \qquad \sigma\begin{pmatrix} A_1 & \boxed{B_1} \\ pC_1 & D_1 \end{pmatrix}$$

$\bar{A}_1 \cdot \bar{B}_2 + \bar{B}_1 \cdot \bar{D}_2 = 0 \quad \because \bar{B}_2 : \text{solvable} \quad \therefore \bar{B}_1 : \text{arbitrary}$

$\bar{C}_1 \cdot \bar{A}_2 + \bar{D}_1 \cdot \bar{C}_2 = 0 \quad \because \bar{C}_1 : \text{solvable} \quad \therefore \bar{C}_2 \ \text{arbitrary}.$

Put $\quad Q_2 = \begin{bmatrix} \bar{A}_2 & 0 \\ \bar{C}_2 & \bar{D}_2 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} \bar{A}_1 & \bar{B}_1 \\ 0 & \bar{D}_1 \end{bmatrix}$

Then $\quad \bar{P}_0' = Q_2 \, \bar{P}_0 \, \sigma(Q_1).$