Theorem[section] [thm]Corollary [thm]Definition

Theory on Well-posedness of Boussinesq equations

3.5 Anisotropic Dissipation Case

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Horizontal dissipation

Consider the IVP for the 2D Boussinesq equations with horizontal dissipation

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p + \nu u_{xx} + \theta \vec{e_2}, \\ \nabla \cdot u = 0, \\ \partial_t \theta + (u \cdot \nabla)\theta = 0, \\ u(x,0) = u_0(x), \ \theta(x,0) = \theta_0(x) \end{cases} \tag{1.1}$$

(1.1) has been shown to possess a unique global solution for suitable (u_0, θ_0) and the following theorem combines the following results:

- Danchin and Paicu, M^3AS (2011).
- Larinos, Lunasin and Titi, JDE (2013).

Theorem 1 (R. Danchin and M. Paicu; Larinos, Lunasin and Titi)

Let $u_0 \in H^1(\mathbb{R}^2)$ *and* $\nabla \cdot u_0 = 0$. *Assume* $\omega_0 = \nabla \times u_0 \in \sqrt{L}$, *namely*

$$\sup_{q \geq 2} \frac{\|\omega_0\|_{L^q}}{\sqrt{q}} < +\infty.$$

Let $\theta_0 \in L^2 \cap L^\infty$. *Then the IVP (1.1) has a unique solution* (u, θ) *satisfying*

$$u \in L^\infty_{loc}([0,\infty); H^1), \quad \omega \in L^\infty_{loc}([0,\infty); \sqrt{L}), \quad u_2 \in L^2_{loc}([0,\infty); H^2),$$

$$\theta \in C_b([0,\infty); L^2), \quad \theta \in L^\infty([0,\infty); L^\infty).$$

Remark: The paper of R. Danchin and M. Paicu (2011) originally assumed that $\theta_0 \in H^s$ with $s \in (1/2, 1)$ to show the uniqueness. Later Larinos, Lunasin and Titi (2013) was able to prove the uniqueness without this assumption.

First, we prove the global existence of weak solutions in a very weak functional setting via Friedriches' Method. This method cuts off the high frequencies and thus smooths the functions. The global existence result can be stated as follows.

Theorem 2 (Global weak solution)

Let $\theta_0 \in L^2 \cap L^\infty$ and $u_0 \in H^1$ and $\nabla \cdot u_0 = 0$. Then (1.1) has a global weak solution (u, θ) satisfying

$$\theta \in L^\infty([0,\infty); L^2 \cap L^\infty),$$

$$u \in L^\infty_{loc}([0,\infty); H^1), \;\; u_2 \in L^2_{loc}([0,\infty); H^2).$$

Proof. (Friedriches' Method) Let $n \in \mathbb{N}$ and define

$$L_n^2 = \{f \in L^2(\mathbb{R}^2) | \operatorname{supp}\widehat{f} \subset B(0,n)\},$$

$$J_n f = (\chi_{B(0,n)}\widehat{f}\,)^{\vee},$$

where $\widehat{f}$ and $f^{\vee}$ denotes the Fourier and the inverse Fourier transforms, respectively, and $\chi_{B(0,n)}$ is the characteristic function on $B(0,n)$. Clearly, $J_n f \in H^{\infty} = \cap_{s\geq 0} H^s$.

Consider the equations

$$\begin{cases} \partial_t \theta + J_n \nabla \cdot (J_n u J_n \theta) = 0, \\ \partial_t u + \mathcal{P} J_n \nabla \cdot (J_n \mathcal{P} u \otimes J_n \mathcal{P} u) = \nu J_n \mathcal{P} \partial_{xx} u + J_n \mathcal{P}(\theta \overrightarrow{e_2}), \\ u(x,0) = J_n u_0, \quad \theta(x,0) = J_n \theta_0, \end{cases} \tag{1.2}$$

where $\mathcal{P}$ denotes the Leray projection. By the Picard theorem, there exists $T^* > 0$ and a solution $(u, \theta) \in C^1([0, T), L^2_n)$

Noticing that $J_n f = f$ if $f \in L^2_n$ and $\mathcal{P}F = F$ if $\nabla \cdot F = 0$, we have

$$\begin{cases} \partial_t \theta + J_n \nabla \cdot (u\theta) = 0, \\ \partial_t u + \mathcal{P} J_n \nabla \cdot (u \otimes u) = \nu \partial_{xx} u + \mathcal{P}(\theta \vec{e_2}). \end{cases}$$

By the energy method

$$\|\theta\|_{L^2} \leq \|J_n \theta_0\|_2 \leq \|\theta_0\|_{L^2},$$

$$\|u\|_{L^2}^2 + 2\nu \int_0^t \|\partial_x u\|_{L^2}^2 dt \leq (\|u_0\|_{L^2} + t\|\theta_0\|_{L^2})^2.$$

Taking the curl of the u-equation yields

$$\partial_t \omega + \mathcal{P} J_n (u \cdot \nabla \omega) = \nu \partial_{xx} \omega + \partial_x \theta$$

and thus

$$\begin{aligned} \|\omega\|_2^2 + 2\nu \int_0^t \|\partial_x \omega\|_2^2 d\tau &\le \int_0^t \|\theta_0\|_2 \|\partial_x \omega\|_2 d\tau \\ &\le \nu \int_0^t \|\partial_x \omega\|_2^2 d\tau + \frac{C}{\nu} \int_0^t \|\theta_0\|_2^2 d\tau. \end{aligned}$$

Therefore, $\theta \in L^\infty([0,\infty); L^2)$ and $u \in L^\infty([0, T]; H^1)$ for any $T > 0$. By the Picard Extension Theorem, (θ, u) is global in time and admits bounds that are uniform in n,

$$\theta^{(n)} \in L^\infty([0,\infty); L^2), \qquad u^{(n)} \in L^\infty([0, T]; H^1).$$

In addition, it can be shown that

$$\partial_t \theta^{(n)} \in L^\infty([0,T]; H^{-3/2}), \qquad \partial_t u^{(n)} \in L^\infty([0,T]; H^{-1})$$

Since $L^2 \hookrightarrow H^{-3/2}$ locally and $H^1 \hookrightarrow H^{-1}$ locally, the Aubin-Lions compactness lemma then implies $u^{(n)} \to u$ in H^L_{loc} for any $-1 \le L < 1$ and $\theta^{(n)} \to \theta$ in H^L_{loc} for any $-3/2 \le L < 0$. We can use these convergence to pass the limit in the weak formulation. This completes the proof. $\square$

Here, we show that (1.1) has a unique local classical solution.

Theorem 1 (Unique local classical solution)

Let $(u_0, \theta_0) \in H^s \times H^{s-1}$ with $s > 2$. Then there is $T > 0$ and a unique solution $(u, \theta) \in C([0, T), H^s \times H^{s-1})$ satisfying (1.1).

Proof. We can either use the mollifier approach as in the book of Majda and Bertozzi or Friedriches' approach above. The crucial part is a local priori bound. Define

$$J^s f = (1 - \Delta)^{s/2} f \quad \text{or} \quad \widehat{J^s f}(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi).$$

Then clearly $\|J^s f\|_{L^2} = \|f\|_{H^s}$.

It follows from the vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega = \nu \partial_{x_1 x_1} \omega + \partial_{x_1} \theta$$

that

$$\frac{1}{2} \frac{d}{dt} \| J^{s-1} \omega \|_2^2 + \nu \| \partial_{x_1} J^{s-1} \omega \|_2^2 = K_1 + K_2,$$

where

$$K_1 = \int \partial_{x_1} J^{s-1} \theta J^{s-1} \omega,$$

$$K_2 = - \int (J^{s-1}(u \cdot \nabla \omega) - u \cdot \nabla J^{s-1} \omega) J^{s-1} \omega.$$

K_1 and K_2 can be bounded as follows. By integration by parts,

$$|K_1| \leq \frac{\nu}{2}\|\partial_{x_1} J^{s-1}\omega\|_2^2 + \frac{1}{2\nu}\|J^{s-1}\theta\|_2^2.$$

Applying the commutator estimates yields

$$\begin{aligned}
|K_2| &\leq \|J^{s-1}(u\cdot\nabla\omega) - u\cdot\nabla J^{s-1}w\|_2\|J^{s-1}\omega\|_2 \\
&\leq C(\|J^s u\|_2\|\omega\|_\infty + \|J^{s-1}\omega\|_2\|\nabla u\|_\infty)\|J^{s-1}\omega\|_2 \\
&\leq C\,\|\nabla u\|_\infty\|J^{s-1}w\|_2^2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|J^{s-1}\theta\|_2^2 \\
= & -\int (J^{s-1}(u\cdot\nabla\theta) - u\cdot\nabla J^{s-1}\theta)J^{s-1}\theta \\
\leq & C\|J^{s-1}\theta\|_2(\|J^s u\|_2\|\theta\|_\infty + \|J^{s-1}\theta\|_2\|\nabla u\|_\infty) \\
\leq & C\|\theta\|_\infty(\|J^{s-1}\omega\|_2^2 + \|J^{s-1}\theta\|_2^2) + C\|\nabla u\|_\infty\|J^{s-1}\theta\|_2^2
\end{aligned}$$

Therefore, $Y(t) = \|J^{s-1}\omega\|_2^2 + \|J^{s-1}\theta\|_2^2$ satisfies

$$\frac{d}{dt}Y(t) \leq C\left(1 + \|\theta\|_\infty + \|\nabla u\|_\infty\right) Y(t).$$

Since $s > 2$,

$$\|\nabla u\|_\infty \leq C\|\nabla u\|_{H^{s-1}} \leq C\|J^{s-1}\omega\|_2.$$

and consequently

$$\frac{d}{dt}Y(t) \leq C\left(1 + \|\theta\|_\infty + \sqrt{Y(t)}\right) Y(t).$$

This inequality implies that $\exists T^*$ such that $Y(t) \leq C$ for $t < T^*$.
This completes the proof. $\square$

Here we establish a global bound for $\|u\|_{H^s}$ and $\|\theta\|_{H^{s-1}}$, which allows us to extend the local solution in the previous subsection to a global one.

Theorem 2 (Global classical solution)

Assume that $(u_0, \theta_0) \in H^s \times H^{s-1}$ with $s > 2$. Then (1.1) has a unique global solution $(u, \theta) \in C([0, \infty), H^s \times H^{s-1})$.

Proof. It suffices to obtain a global bound for (u, θ) in $H^s \times H^{s-1}$. From the previous proof, $Y(t) = \|J^{s-1}\omega\|_2^2 + \|J^{s-1}\theta\|_2^2$ satisfies

$$\frac{d}{dt} Y(t) \leq C(1 + \|\theta\|_\infty + \|\nabla u\|_\infty)\, Y(t). \tag{1.3}$$

The trick is still to control $\|\nabla u\|_\infty$ through the following logarithmic interpolation inequality,for $\sigma > 1$,

$$\|f\|_\infty \leq \sup_{q \geq 2} \frac{\|f\|_{L^q}}{\sqrt{q}} \sqrt{\log(1 + \|f\|_{H^\sigma})}.$$

which proof gives later.

In particular, for $s > 2$, we have

$$\begin{aligned} \|\nabla u\|_\infty &\leq C \sup_{q\geq 2} \frac{\|\nabla u\|_q}{\sqrt{q}} \sqrt{\log(1 + \|\nabla u\|_{H^{s-1}})} \\ &\leq C \sup_{q\geq 2} \frac{\|\nabla u\|_q}{\sqrt{q}} \sqrt{\log(1 + \|\omega\|_{H^{s-1}})}. \end{aligned} \tag{1.4}$$

The goal is then to show that, for any $T > 0$,

$$\int_0^T \sup_{q\geq 2} \frac{\|\nabla u\|_q}{\sqrt{q}}\, dt < \infty. \tag{1.5}$$

This is accomplished through two steps. First, we have

$$\|\omega(T)\|_2^2 + 2\nu \int_0^T \|\partial_{x_1}\omega\|_2^2 \, dt \le \frac{2T}{\nu}\|\theta_0\|_2^2$$

and, from the L^q estimate of ω, $2 \le q < \infty$,

$$\|\omega\|_q^2 \le \|\omega_0\|_q^2 + \frac{2(q-1)}{\nu} t \|\theta_0\|_q^2$$

$$\|\omega\|_q \le \|\omega_0\|_q + \sqrt{\frac{2}{\nu} t(q-1)} \|\theta_0\|_q.$$

Thus, we have

$$\sup_{q \ge 2} \frac{\|\omega\|_q}{\sqrt{q}} \le \sup_{q \ge 2} \frac{\|\omega_0\|_q}{\sqrt{q}} + \sqrt{\frac{2t}{\nu}} \|\theta_0\|_{L^2 \cap L^\infty}.$$

We caution that the inequality $\|\nabla u\|_q \le \frac{c\,q^2}{q-1}\|\omega\|_q$ does not really help. Instead, we use the imbedding inequality and prove in the following Lemma, for a constant C independent of q,

$$\sup_{q\ge 2} \frac{\|f\|_q}{\sqrt{q}} \le C\,\|f\|_{H^1}$$

and the simple fact that $\|\partial_{x_1}\omega\|_2 = \|\nabla\partial_{x_1}u\|_2$, we have

$$\begin{aligned}\int_0^T \sup_{q\ge 2} \frac{\|\partial_{x_1}u\|_q}{\sqrt{q}}\,dt &\le C\int_0^T \|\partial_{x_1}\nabla u\|_2 dt\\ &= C\int_0^T \|\partial_{x_1}\omega\|_2 dt < \infty.\end{aligned}$$

Due to the divergence-free condition $\partial_{x_1} u_1 + \partial_{x_2} u_2 = 0$, we also have

$$\int_0^T \sup_{q\geq 2} \frac{\|\partial_{x_2} u_2\|_q}{\sqrt{q}}\, dt < \infty.$$

In addition, $\partial_{x_2} u_1 = \partial_{x_1} u_2 - \omega$ and thus

$$\int_0^T \sup_{q\geq 2} \frac{\|\partial_{x_2} u_1\|_q}{\sqrt{q}}\, dt \leq \int_0^T \sup_{q\geq 2} \frac{\|\partial_{x_1} u_2\|_q}{\sqrt{q}}\, dt + \int_0^T \sup_{q\geq 2} \frac{\|\omega\|_q}{\sqrt{q}}\, dt < \infty.$$

Thus we have proven (1.5). Combining (1.3), (1.4) and (1.5) yields the desired global bound. This completes the proof of Theorem 2. $\square$

Lemma 3

For $\sigma > 1$,

$$\|f\|_\infty \le C(\sigma) \sup_{q\ge 2} \frac{\|f\|_{L^q}}{\sqrt{q}} \sqrt{\log\left(1+\|f\|_{H^\sigma}\right)}, \tag{1.6}$$

$$\sup_{q\ge 2} \frac{\|f\|_{L^q}}{\sqrt{q}} \le C\|f\|_{H^1}, \tag{1.7}$$

where C*'s are constants.*

Proof. For proving (1.6), one may split f into low and high frequencies according to the Littlewood-Paley decomposition. More precisely, for any $q \in \mathbb{N}$ one may write

$$f = S_q f + \sum_{p \geq q} \Delta_p f.$$

We thus have

$$\|f\|_{L^\infty} \leq \|S_q f\|_{L^\infty} + \sum_{p \geq q} \|\Delta_p f\|_{L^\infty},$$

whence, using the Bernstein inequalities,

$$\|f\|_{L^\infty} \leq C \frac{\|f\|_{L^q}}{\sqrt{q}} \sqrt{q} + C \sum_{p \geq q} 2^p \|\Delta_p f\|_{L^2}.$$

For simplicity, we set $\|f\|_{\sqrt{L}} := \sup_{q\geq 2} \frac{\|f\|_{L^q}}{\sqrt{q}}$. By $\sigma > 1$, we get

$$\|f\|_{L^\infty} \leq C\|f\|_{\sqrt{L}}\sqrt{q} + C2^{q(1-\sigma)}\|f\|_{H^\sigma}.$$

Now, if $C\|f\|_{H^\sigma} \leq \|f\|_{\sqrt{L}}\sqrt{\log(1+\|f\|_{H^\sigma})}$, then take q the integer part of $\log(1+\|f\|_{H^\sigma})$. Otherwise, one may choose for q the integer part of

$$\frac{1}{\sigma-1}\log_2\left(\frac{C\|f\|_{H^\sigma}}{\|f\|_{\sqrt{L}}\sqrt{\log(1+\|f\|_{H^\sigma})}}\right),$$

and we get the desired result (1.6).

For any $q \in [2, \infty)$ and $f \in H^1$, using the Littlewood-Paley decomposition and a Bernstein inequality enables us to write

$$\begin{aligned}
\|f\|_{L^q} &\leq \sum_{p \geq -1} \|\Delta_p v\|_{L^q} \\
&\leq C \sum_{p \geq -1} 2^{-2p/q} 2^p \|\Delta_p v\|_{L^2} \\
&\leq C \left(\sum_{p \geq -1} 2^{-4p/q} \right)^{1/2} \|f\|_{H^1} \\
&\leq C \sqrt{q-1} \|f\|_{H^1},
\end{aligned}$$

which completes the proof of inequality (1.7). $\square$

Here we outlines the proof of Theorem 1. It consists of two main steps. The first step proves the existence, the second proves the uniqueness.

To prove the existence, we first regularize the initial condition. For $\epsilon > 0$, set

$$J_\epsilon f = \rho_\epsilon * f$$

where $\rho_\epsilon = \frac{1}{\epsilon^2}\rho_0(\frac{x}{\epsilon})$ with $\rho_0 \in C_0^\infty(\mathbb{R}^2)$, $\int_{\mathbb{R}^2} \rho_0(x)\, dx = 1$ and

$$\rho(x) = \begin{cases} 1, & \text{if} \quad |x| \le \frac{1}{2}, \\ 0, & \text{if} \quad |x| > 1. \end{cases}$$

Consider (1.1) with the initial data $u^\epsilon(x,0) = J_\epsilon * u_0$ and $\theta^\epsilon(x,0) = J_\epsilon * \theta_0$. Since, for any $s > 0$,

$$\|u_0^\epsilon\|_{H^s} \leq C\frac{1}{\epsilon^{s-1}}\|u_0\|_{H^1}, \quad \|\theta_0^\epsilon\|_{H^s} \leq C\frac{1}{\epsilon^s}\|\theta_0\|_{L^2},$$

where C is a constant independent of ϵ. By Theorem 2, there exists a unique solution

$$(u^\epsilon, \theta^\epsilon) \in C([0,\infty); H^s \times H^{s-1}).$$

Since $(u^\epsilon, \theta^\epsilon)$ admits a global uniform bound in $H^1 \times L^2$, $(u^\epsilon, \theta^\epsilon) \rightharpoonup (u, \theta)$ in $H^1 \times L^2$ and (u, θ) is a weak solution according to Theorem 2.

Uniqueness.

Danchin and Paicu (2011) assumed that

$$\theta_0 \in H^s \cap L^\infty \quad \text{with} \quad s \in \left(\frac{1}{2}, 1\right).$$

Larios, Lunasin and Titi (2013) assumed that

$$\theta_0 \in L^2 \cap L^\infty.$$

The essential idea is the Yudovich approach. Let $(u^{(1)}, \theta^{(1)})$ and $(u^{(2)}, \theta^{(2)})$ be two solutions. Then the difference $(\widetilde{u}, \widetilde{\theta})$,

$$\widetilde{u} = u^{(1)} - u^{(2)}, \quad \widetilde{\theta} = \theta^{(1)} - \theta^{(2)},$$

satisfies

$$\begin{cases} \partial_t \widetilde{u} + u^{(1)} \cdot \nabla \widetilde{u} + \widetilde{u} \cdot \nabla u^{(2)} = \partial_{xx} \widetilde{u} - \nabla \widetilde{p} + \widetilde{\theta} \vec{e}_2, \\ \partial_t \widetilde{\theta} + u^{(1)} \cdot \nabla \widetilde{\theta} + \widetilde{u} \cdot \nabla \theta^{(2)} = 0. \end{cases}$$

The second equation demands too much regularity on θ. To exchange into a variable with less regularity requirement, we introduce

$$\Delta\xi^{(1)} = \theta^{(1)}, \quad \Delta\xi^{(2)} = \theta^{(2)}, \quad \widetilde{\xi} = \xi^{(1)} - \xi^{(2)}.$$

Clearly, $\widetilde{\xi}$ satisfies

$$\partial_t \Delta\widetilde{\xi} + u^{(1)} \cdot \nabla(\Delta\widetilde{\xi}) + \widetilde{u} \cdot \nabla\Delta\xi^{(2)} = 0.$$

The goal is to show $\widetilde{u} \equiv 0$ and $\widetilde{\xi} \equiv 0$.

It follows from simple energy estimates that

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{u}\|_2^2 + \|\partial_x \widetilde{u}\|_2^2 = -\int \widetilde{u}\cdot\nabla u^{(2)}\cdot\widetilde{u} + \int \Delta\widetilde{\xi}\vec{e}_2\cdot\widetilde{u},$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla\widetilde{\xi}\|_2^2 = -\int u^{(1)}\cdot\nabla\Delta\widetilde{\xi}\,\widetilde{\xi} - \int \widetilde{u}\cdot\nabla\Delta\xi^{(2)}\,\widetilde{\xi}.$$

We now estimates the terms on the right.

$$\begin{aligned}\int \Delta\widetilde{\xi}\,\vec{e}_2\cdot\widetilde{u} &= \int \Delta\widetilde{\xi}\,\widetilde{u}_2 = \int (\partial_{x_1x_1} + \partial_{x_2x_2})\widetilde{\xi}\,\widetilde{u}_2 \\ &= -\int \partial_{x_1}\widetilde{\xi}\partial_{x_1}\widetilde{u}_2 + \int \partial_{x_2}\widetilde{\xi}\partial_{x_1}\widetilde{u}_1 \\ &\le \frac{1}{2}\|\nabla\widetilde{\xi}\|_2^2 + \frac{1}{2}\|\partial_{x_1}\widetilde{u}\|_2^2.\end{aligned}$$

Integrating by parts and noticing that $\theta^{(2)} = \Delta\xi^{(2)}$, we have

$$\begin{aligned} -\int \widetilde{u}\cdot\nabla\Delta\xi^{(2)}\widetilde{\xi} &= \int \widetilde{u}\cdot\nabla\widetilde{\xi}\Delta\xi^{(2)} \\ &\le \frac{1}{2}\|\theta^{(2)}\|_\infty(\|\widetilde{u}\|_2^2 + \|\nabla\widetilde{\xi}\|_2^2). \end{aligned}$$

For notational convenience, we write

$$J_1 = -\int \widetilde{u}\cdot\nabla u^{(2)}\cdot\widetilde{u}, \qquad J_2 = -\int u^{(1)}\cdot\nabla\Delta\widetilde{\xi}\ \widetilde{\xi}.$$

Clearly

$$|J_1| \le \|\widetilde{u}\|_\infty^{\frac{2}{p}}\int|\nabla u^{(2)}||\widetilde{u}|^{2-\frac{2}{p}} \le \|\widetilde{u}\|_\infty^{\frac{2}{p}}\|\nabla u^{(2)}\|_p\|\widetilde{u}\|_2^{2-\frac{2}{p}}$$

We know that

$$\sup_{p\geq 2} \frac{\|\nabla u^{(2)}\|_p}{p} \leq C_0$$

for C_0 independent of p. Also we have $\|\widetilde{u}\|_\infty \leq M \equiv C\,\|\widetilde{u}\|_2^a\,\|\nabla\widetilde{u}\|_q^{1-a}$ for any $q>2$ and $a=\frac{q-2}{2(q-1)}$. Therefore,

$$|J_1| \leq C_0\,p\,M^{\frac{2}{p}}\,\|\widetilde{u}\|_2^{2-\frac{2}{p}}.$$

Using the simple fact that a function $f(x)=xA^{\frac{2}{x}}$ has the minimum $2e\log A$, we have

$$|J_1| \leq 2C_0\,(\log M - \log\|\widetilde{u}\|_2)\|\widetilde{u}\|_2^2.$$

Integrating by parts yields

$$
\begin{aligned}
J_2 &= \int u^{(1)} \cdot \nabla\widetilde{\xi}\, \Delta\widetilde{\xi} \\
&= \int u^{(1)} \cdot \nabla\widetilde{\xi}\, \partial_k\partial_k\widetilde{\xi} \\
&= -\int \partial_k u^{(1)} \cdot \nabla\widetilde{\xi}\, \partial_k\widetilde{\xi} - \int u^{(1)} \cdot \nabla\partial_k\widetilde{\xi}\, \partial_k\widetilde{\xi} \\
&= -\int \partial_k u^{(1)} \cdot \nabla\widetilde{\xi}\, \partial_k\widetilde{\xi}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|J_2| &\le \int |\nabla u^{(1)}|\, |\nabla\widetilde{\xi}|^2 \\
&\le \|\nabla\widetilde{\xi}\|_\infty^{\frac{2}{p}} \int |\nabla u^{(1)}| |\nabla\widetilde{\xi}|^{2-\frac{2}{p}}.
\end{aligned}
$$

Since $\|\nabla\widetilde{\xi}\|_\infty \leq M$,

$$
\begin{aligned}
|J_2| &\leq CpM^{\frac{2}{p}}\|\nabla\widetilde{\xi}\|_2^{2-\frac{2}{p}} \\
&\leq C(\log M - \log\|\nabla\widetilde{\xi}\|_2)\,\|\nabla\widetilde{\xi}\|_2^2.
\end{aligned}
$$

Combining the estimates allows us to conclude that $X(t) = \|\widetilde{u}\|_2^2 + \|\nabla\widetilde{\xi}\|_2^2$ satisfies

$$
\frac{d}{dt}X + \|\partial_x\widetilde{u}\|_2^2 \leq CX + C\,(\log M - \log X)\,X,
$$

where C's are constants.

$$
\begin{aligned}
\frac{d}{dt}X - C\,X &\leq C\,(\log M - \log X)X \\
\frac{d}{dt}(e^{-C\,t}X) &\leq Ce^{-C\,t}(\log M - \log X)X \\
X(t) &\leq e^{C\,t}X(0) + C\int_0^t e^{C\,(t-\tau)}(\log M - \log X)X\,d\tau.
\end{aligned}
$$

Since $X(0) = 0$, we have $X(t) \equiv 0$ for any $t > 0$ by Osgood inequality (see Lemma 4 below). This completes the proof of Theorem 1. $\square$

Lemma 4 (Osgood Inequality)

Let $\rho_0 \geq 0$ *be a constant and* $\alpha(t) \geq 0$ *be a continuous function. Assume*

$$\rho(t) \leq \rho_0 + \int_0^t \alpha(\tau) w(\rho(\tau)) d\tau,$$

where w *satisfies*

$$\int_1^{\infty} \frac{1}{w(r)} dr = \infty.$$

Then $\rho_0 = 0$ *implies* $\rho \equiv 0$, *and* $\rho_0 > 0$ *implies that*

$$-\Omega(\rho(t)) + \Omega(\rho_0) \leq \int_0^t \alpha(\tau) d\tau,$$

where $\Omega(\rho) = \int_\rho^1 \frac{dr}{w(r)}$.

In particular, the Osgood inequality applies to $w(\rho) = \rho \log \frac{M}{\rho}$ since

$$\int_1^\infty \frac{1}{r(\log M - \log r)} dr = \infty.$$

Horizontal thermal diffusion

We work on the 2D Boussinesq equations with only horizontal diffusion

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p + \theta \vec{e_2}, \\ \partial_t \theta + (u \cdot \nabla)\theta = \partial_{x_1 x_1} \theta, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.8}$$

where we have set the coefficient of $\partial_{x_1 x_1}\theta$ to be 1, without loss of generality. (1.8) still possesses a unique global solution when (u_0, θ_0) is in a suitable functional setting. The following Theorem is coming from the following reference:

- Danchin and Paicu, M^3AS (2011).

Theorem 3

Assume $u_0 \in H^1, \omega_0 = \nabla \times u_0 \in L^\infty, \theta_0 \in H^1 \cap L^\infty, |\partial_{x_1}|^{1+s}\theta_0 \in L^2$ *for* $s \in (0, \frac{1}{2}]$. *Then (1.8) has a unique global solution* (u, θ) *satisfying*

$$u \in C([0,\infty); H^1), \quad \omega \in L^\infty_{loc}([0,\infty); L^\infty),$$

$$\theta \in C([0,\infty); H^1 \cap L^\infty),$$

$$|\partial_{x_1}|^{1+s}\theta \in L^\infty_{loc}([0,\infty); L^2), \quad |\partial_{x_1}|^{2+s}\theta \in L^2_{loc}([0,\infty); L^2).$$

Here $|\partial_{x_1}|^\beta$ with $\beta \in \mathbb{R}$ is defined in terms of its Fourier transform,

$$|\partial_{x_1}|^\beta f(x) = \int e^{ix\cdot\xi}|\xi_1|^\beta \hat{f}(\xi_1, \xi_2)d\xi_1 d\xi_2.$$

The equations give us for free the global bounds on $\|\theta\|_{L^2\cap L^\infty}$ and $\|u\|_{H^1}$. The first real step is to obtain a global bound for $\|\nabla\theta\|_2$. This can be established by writing the nonlinear term explicitly in terms of the partial derivatives in different directions and fully take advantage of the dissipation in the x-direction. The resulting estimate is given by

$$\frac{d}{dt}\|\nabla\theta\|_{L^2}^2 + \|\partial_{x_1}\nabla\theta\|_{L^2}^2 \leq B(t)\|\nabla\theta\|_{L^2}^2,$$

where $B(t)$ is integrable on $[0,\infty)$. Some special consequences of this inequality are

$$\|\omega(\cdot,T)\|_{L^4} \leq C(T), \quad \|u(\cdot,T)\|_{L^\infty} \leq C(T),$$

where $T > 0$ is an arbitrarily fixed and $C(T)$ depends on the initial data and T.

In fact, by the Sobolev embedding inequality

$$\|\partial_{x_1}\theta\|_{L^4} \le C\|\partial_{x_1}\theta\|_{L^2}^{1/2}\|\nabla\partial_{x_1}\theta\|_{L^2}^{1/2},$$

we have

$$\int_0^T \|\partial_{x_1}\theta\|_{L^4}^4\, dt \le C \sup_{t\in[0,T]} \|\partial_{x_1}\theta\|_{L^2}^2 \int_0^T \|\partial_{x_1}\nabla\theta\|_{L^2}^2\, dt < \infty.$$

It then follows from the vorticity equation that

$$\frac{1}{4}\frac{d}{dt}\|\omega\|_{L^4}^4 = \int \partial_{x_1}\theta\omega|\omega|^2 dx \le \|\partial_{x_1}\theta\|_{L^4}\|\omega\|_{L^4}^3.$$

Therefore,

$$\begin{aligned} \|\omega\|_{L^4} &\leq \|\omega_0\|_{L^4} + \int_0^t \|\partial_{x_1}\theta\|_{L^4} d\tau \\ &\leq \|\omega_0\|_{L^4} + \left(\int_0^t \|\partial_{x_1}\theta\|_{L^4}^4 d\tau\right)^{1/4} t^{3/4}. \end{aligned}$$

Thus,

$$\|u\|_{L^\infty} \leq \|u\|_{L^2}^{1/3} \|\nabla u\|_{L^4}^{2/3} \leq C\, \|u\|_{L^2}^{1/3} \|\omega\|_{L^4}^{2/3} < \infty.$$

Next we show that, for $s \in (0, 1/2]$,

$$|\partial_{x_1}|^{1+s}\theta \in L^\infty_{loc}([0,\infty); L^2), \quad |\partial_{x_1}|^{2+s}\theta \in L^2_{loc}([0,\infty); L^2). \quad (1.9)$$

Clearly, $|\partial_{x_1}|^{1+s}\theta$ satisfies

$$\partial_t |\partial_{x_1}|^{1+s}\theta - \partial_{x_1x_1}|\partial_{x_1}|^{1+s}\theta = -|\partial_{x_1}|^{1+s}(u\cdot\nabla\theta).$$

Taking the inner product with $|\partial_{x_1}^{1+s}|\theta$, we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\||\partial_{x_1}|^{1+s}\theta\|_{L^2}^2 + \||\partial_{x_1}|^{2+s}\theta\|_{L^2}^2 \\
&= -\int |\partial_{x_1}|^{1+s}(u\cdot\nabla\theta)\,|\partial_{x_1}|^{1+s}\theta\,dx \\
&= -\int |\partial_{x_1}|(u\cdot\nabla\theta)\,|\partial_{x_1}|^{1+2s}\theta\,dx \\
&= \int \frac{\partial_{x_1}}{|\partial_{x_1}|}(\partial_{x_1}u\cdot\nabla\theta + u\cdot\nabla\partial_{x_1}\theta)|\partial_{x_1}|^{1+2s}\theta dx.
\end{aligned}$$

Since $s \in (0, 1/2]$,

$$\begin{aligned} & \int \frac{\partial_{x_1}}{|\partial_{x_1}|}(u \cdot \nabla \partial_{x_1}\theta)|\partial_{x_1}|^{1+2s}\theta dx \\ \leq \quad & \|u\|_{L^\infty}\|\nabla\partial_{x_1}\theta\|_{L^2}\||\partial_{x_1}|^{1+2s}\theta\|_{L^2} \\ \leq \quad & \|u\|_{L^\infty}\|\partial_{x_1}\nabla\theta\|_{L^2}^2. \end{aligned}$$

Writing $\partial_{x_1}u \cdot \nabla\theta = \partial_{x_1}u_1\partial_{x_1}\theta + \partial_{x_1}u_2\partial_{x_2}\theta$, we have

$$\begin{aligned} \int \frac{\partial_{x_1}}{|\partial_{x_1}|}(\partial_{x_1}u \cdot \nabla\theta)\,|\partial_{x_1}|^{1+2s}\theta dx \quad \leq \quad & \|\omega\|_{L^4}\|\partial_{x_1}\theta\|_{L^4}\||\partial_{x_1}|^{1+2s}\theta\|_{L^2} \\ & + \int \frac{\partial_{x_1}}{|\partial_{x_1}|}\partial_{x_1}u_2\partial_{x_2}\theta|\partial_{x_1}|^{1+2s}\theta. \end{aligned}$$

In addition,

$$\begin{aligned}
\int \frac{\partial_{x_1}}{|\partial_{x_1}|}\partial_{x_1} u_2 \partial_{x_2}\theta |\partial_{x_1}|^{1+2s}\theta &= -\int \frac{\partial_{x_1}}{|\partial_{x_1}|} u_2 \partial_{x_1}\partial_{x_2}\theta |\partial_{x_1}|^{1+2s}\theta dx \\
&\quad -\int \frac{\partial_{x_1}}{|\partial_{x_1}|} |\partial_{x_1}|^{s}(u_2\partial_{x_2}\theta)|\partial_{x_1}|^{2+s}\theta dx \\
&\leq \|u_2\|_{L^\infty}\|\partial_{x_1}\nabla\theta\|_{L^2}\||\partial_{x_1}|^{1+2s}\theta\|_{L^2} \\
&\quad +\||\partial_{x_1}|^{s}(u_2\partial_{x_2}\theta)\|_{L^2}\||\partial_{x_1}|^{2+s}\theta\|_{L^2} \\
&\leq \|u_2\|_{L^\infty}^2\|\partial_{x_1}\nabla\theta\|_{L^2}^2 + \frac{1}{4}\||\partial_{x_1}|^{2+s}\theta\|_{L^2}^2 \\
&\quad + C\|u\|_{H^1}^2(\|\partial_2\theta\|_{L^2} + \|\partial_1\partial_2\theta\|_{L^2})^2,
\end{aligned}$$

the last inequality have used the following inequality

$$\|fg\|_{L^2_{x_2}(H^{1/2}_{x_1})} \leq C\|f\|_{H^1}(\|g\|_{L^2} + \|\partial_1 g\|_{L^2}).$$

Combining the estimates yield, for some $g \in L^1_{loc}([0,\infty))$,

$$\frac{d}{dt}\||\partial_{x_1}|^{1+s}\theta\|^2_{L^2} + \||\partial_{x_1}|^{2+s}\theta\|^2_{L^2} \le g(t).$$

We thus have obtained (1.9). A special consequence is that

$$\omega \in L^\infty_{loc}([0,\infty); L^\infty).$$

This can be obtained by combining

$$\|\omega\|_{L^\infty} \le \|\omega_0\|_{L^\infty} + \int_0^t \|\partial_{x_1}\theta\|_{L^\infty} d\tau$$

with the simple estimate of the following lemma. ‘

Lemma 4

If $\frac{1}{s_1} + \frac{1}{s_2} < 2, \quad s_1 > 0, \quad s_2 > 0,$ *then*

$$\|f\|_{L^\infty} \leq C(\|f\|_{L^2} + \||\partial_{x_1}|^{s_1} f\|_{L^2} + \||\partial_{x_2}|^{s_2} f\|_{L^2}).$$

Applying this lemma with $s_1 = 1 + s, \quad s_2 = 1$, we have

$$\|\omega\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} + C \int_0^t (\|\partial_{x_1}\theta\|_{L^2} + \||\partial_{x_1}|^{2+s}\theta\|_{L^2} + \|\partial_{x_1}\partial_{x_2}\theta\|_{L^2}) d\tau < \infty.$$

Trivially, interpolating between L^2 and L^∞ yields $\omega \in L^q$ for $q \in [2, \infty)$. More importantly,

$$\nabla u \in L^\infty_{loc}([0,\infty); L) \quad \text{or} \quad \sup_{t \in [0,T]} \sup_{q \geq 2} \frac{\|\nabla u\|_{L^q}}{q} \leq C(T).$$

This completes the part for the existence and regularity part.

Finally we show the uniqueness. This is a consequence of the Yudovich type argument. Let $(u^{(1)}, \theta^{(1)})$ and $(u^{(2)}, \theta^{(2)})$ be two solutions. Then $\tilde{u} = u^{(1)} - u^{(2)}$ and $\tilde{\theta} = \theta^{(1)} - \theta^{(2)}$ satisfy

$$\begin{cases} \partial_t \tilde{u} + u^{(1)} \cdot \nabla \tilde{u} + \tilde{u} \cdot \nabla u^{(2)} = \theta \overrightarrow{e_2}, \\ \partial_t \tilde{\theta} + u^{(1)} \cdot \nabla \tilde{\theta} + \tilde{u} \cdot \nabla \theta^{(2)} = \partial_{x_1 x_1} \tilde{\theta}. \end{cases}$$

The most difficult term we would encounter in the further estimates is

$$\int \tilde{u_2}\, \partial_{x_2} \theta^{(2)}\, \tilde{\theta} dx.$$

One way to handle it is to use the divergence free $\mathrm{div}\, u = 0$ and the identity

$$\tilde{u}_2 = (I - \partial_{x_2}^2)^{-1}\tilde{u}_2 - (I - \partial_{x_2}^2)^{-1}\partial_{x_2}\partial_{x_1}\tilde{u}_1.$$

We shall omit further details. This completes the proof of Theorem 3.

Vertical dissipation and vertical thermal diffusion

2D Boussinesq equations with vertical viscosity and vertical thermal diffusion as follows:

$$\begin{cases} \partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} = -\nabla p + \nu \partial_{yy} \vec{u} + \theta \overrightarrow{e_2} \\ \partial_t \theta + \vec{u} \cdot \nabla \theta = \kappa \partial_{yy} \theta \\ \nabla \cdot \vec{u} = 0 \\ \vec{u}(x, 0) = \vec{u}_0(x), \quad \theta(x, 0) = \theta_0(x), \end{cases} \tag{1.10}$$

here $\vec{u} = (u,\ v)$.

The global existence and uniqueness of (1.10) were established in

- Cao Chongsheng, Wu Jiahong, Global regularity for the 2D anisotropic Boussinesq equations with vertical dissipation. ARMA.(2013).

The main results can be stated as follows.

Theorem 5 (Cao, Chongsheng; Wu, Jiahong, 2013)

Consider the IVP for the anisotropic Boussinesq equations with vertical dissipation (1.10). Let $\nu > 0$ *and* $\kappa > 0$. *Let* $(u_0, v_0, \theta_0) \in H^2(\mathbb{R}^2)$. *Then, for any* $T > 0$, *(1.10) has a unique classical solution* (u, v, θ) *on* $[0, T]$ *satisfying*

$$(u, v, \theta) \in C([0, T]; H^2(\mathbb{R}^2)).$$

As we mentioned before, the local existence and uniqueness is not very hard to obtain. Therefore, our effort will be devoted to proving the global *a priori* H^2 bounds for the solution.

- $\|(u, \theta)\|_{L^2}$ estimates.

$$\|\theta(t)\|_2^2 + 2\kappa \int_0^T \|\partial_y \theta\|_2^2 \, dt \le \|\theta_0\|_2^2,$$

$$\|u(t)\|_2^2 + 2\nu \int_0^T \|\partial_y u\|_2^2 \, dt \le (\|u_0\|_2 + t\,\|\theta_0\|_2)^2.$$

- $\|(u, \theta)\|_{H^1}$ estimates ??

Consider the equation for $\omega = \vec{\nabla} \times \vec{u}$, which satisfies

$$\partial_t \omega + \vec{u} \cdot \nabla \omega = \nu \partial_{yy} \omega + \partial_x \theta.$$

Then we have

$$\frac{1}{2}\frac{d}{dt}\int \omega^2 + \nu \int (\partial_y \omega)^2 = \int \partial_x \theta \omega,$$

but we cannot control the $\partial_x \theta$. Therefore, if we want to obtain a global bound for $\|\omega\|_2$, then we need to combine it with the estimate of $\nabla\theta$.

$$\frac{1}{2}\frac{d}{dt}\int |\nabla\theta|^2 + \kappa \int |\partial_y \nabla\theta|^2 = -\int \nabla\theta \cdot \nabla\vec{u} \cdot \nabla\theta. \qquad (1.11)$$

To make use of the dissipation in the y-direction, we write

$$\begin{aligned}\nabla\theta\cdot\nabla\vec{u}\cdot\nabla\theta =& \partial_x u(\partial_x\theta)^2+\partial_x v\partial_x\theta\partial_y\theta \\ &+\partial_y u\partial_x\theta\partial_y\theta+\partial_y v(\partial_y\theta)^2.\end{aligned}$$

To bound the terms on the right, we need the following lemma.

Lemma 5

Assume that $f, g, g_y, h, h_x \in L^2(\mathbb{R}^2)$. Then

$$\iint |f\,g\,h|\,dxdy \le C\,\|f\|_2\,\|g\|_2^{\frac{1}{2}}\,\|g_y\|_2^{\frac{1}{2}}\,\|h\|_2^{\frac{1}{2}}\,\|h_x\|_2^{\frac{1}{2}}. \qquad (1.12)$$

This lemma allows us to bound some of the terms suitably. For example,

$$
\begin{aligned}
\left|\int \partial_y u \partial_x \theta \partial_y \theta\right| &\le C\,\|\partial_y u\|_2\,\|\partial_x \theta\|_2^{\frac{1}{2}}\,\|\|\partial_{xy}\theta\|_2^{\frac{1}{2}}\,\|\partial_y \theta\|_2^{\frac{1}{2}}\,\|\partial_{xy}\theta\|_2^{\frac{1}{2}} \\
&\le \frac{\kappa}{4}\|\partial_{xy}\theta\|_2^2 + C(\kappa)\,\|\partial_y u\|_2^2\,\|\partial_x \theta\|_2\,\|\partial_y \theta\|_2.
\end{aligned}
$$

Integrating by parts, we have

$$
\begin{aligned}
\int \partial_x v \partial_x \theta \partial_y \theta &= -\int \theta\,(\partial_x v \partial_{xy}\theta + \partial_{xy} v \partial_x \theta) \\
&\le \|\theta_0\|_\infty\,\|\partial_x v\|_2\,\|\partial_{xy}\theta\|_2 + \|\theta_0\|_\infty\,\|\partial_x \theta\|_2\,\|\partial_{xy} v\|_2
\end{aligned}
$$

However, the term $\int \partial_x u\,(\partial_x\theta)^2$ can not be bounded suitably. But if we know

$$\int_0^T \|v(t)\|_{L^\infty}^2\,dt < \infty, \tag{1.13}$$

then we have, after integration by parts,

$$\begin{aligned}\int \partial_x u\,(\partial_x\theta)^2 &= -\int \partial_y v\,(\partial_x\theta)^2 = \int v\partial_x\theta\,\partial_{xy}\theta\\ &\le \frac{\kappa}{4}\|\partial_{xy}\theta\|_2^2 + C(\kappa)\,\|v(t)\|_{L^\infty}^2\,\|\partial_x\theta\|_2^2.\end{aligned}$$

Inserting the estimates above in (53) and (1.11), we are able to conclude that, if (1.13) holds, then

$$\|\omega\|_2^2 + \|\nabla\theta\|_2^2 + \nu \int (\partial_y \omega)^2 + \kappa \int |\partial_y \nabla\theta|^2 \le C(T).$$

Unfortunately it appears to be extremely hard to prove (1.13). Therefore, we have to solve this problem through a different route.

This section presents the key ingredients in the proof as well as the proof of Theorem 5.

Proposition 6

Assume $(u_0, v_0, \theta_0) \in H^2$. Let (u, v, θ) be the corresponding classical solution of (1.10). Then the quantity

$$Y(t) = \|\omega\|_{H^1}^2 + \|\theta\|_{H^2}^2 + \|\omega^2 + |\nabla\theta|^2\|_2^2$$

satisfies

$$\begin{aligned}\frac{d}{dt}Y(t) + \|\omega_y\|_{H^1}^2 + \|\theta_y\|_{H^2}^2 + \int \left(\omega^2 + |\nabla\theta|^2\right)\left(\omega^2 + |\nabla\theta_y|^2\right) \\ \leq C\left(1 + \|\theta_0\|_\infty^2 + \|v\|_\infty^2 + \|u_y\|_2^2 + (1 + \|u\|_2^2)\|v_y\|_2^2\right) Y(t),\end{aligned}$$

where C is a constant.

As a special consequence of this differential inequality, we conclude that, if

$$\int_0^T \|v(t)\|_{L^\infty}^2 \, dt < \infty,$$

then $Y(t) < +\infty$ on $[0, T]$.

Proposition 7

Let $(u_0, v_0, \theta_0) \in H^2(\mathbb{R}^2)$ *and let* (u, v, θ) *be the corresponding classical solution of (1.10). Then,*

$$\sup_{r\geq 2} \frac{\|v(t)\|_{L^{2r}}}{\sqrt{r \log r}} \leq \sup_{r\geq 2} \frac{\|v_0\|_{L^{2r}}}{\sqrt{r \log r}} + B(t), \tag{1.14}$$

where $B(t)$ *is an explicit integrable function of* $t \in [0, \infty)$ *that depends on* ν, κ *and the initial norm* $\|(u_0, v_0, \theta_0)\|_{H^2}$*.*

.

Proposition 8 (LogLog type Sobolev inequality)

Let $s > 1$ and $f \in H^s(\mathbb{R}^2)$. Assume that

$$\sup_{r \geq 2} \frac{\|f\|_r}{\sqrt{r \log r}} < \infty.$$

Then there exists a constant C depending on s only such that

$$\|f\|_{L^\infty(\mathbb{R}^2)} \leq C \sup_{r \geq 2} \frac{\|f\|_r}{\sqrt{r \log r}} \left[\log(e + \|f\|_{H^s(\mathbb{R}^2)}) \log\log(e + \|f\|_{H^s(\mathbb{R}^2)})\right]^{\frac{1}{2}}. \tag{1.15}$$

Proof of Theorem 5: Applying Proposition 7 , 8 and using the simple fact that $\|v\|_{H^2}^2 \leq \|\omega\|_{H^1}^2 \leq Y(t)$, we obtain

$$\frac{d}{dt} Y(t) \leq A(t) Y(t) + C\, B^2(t)\, Y(t) \,\log(e + Y(t)) \,\log\log(e + Y(t)),$$

where $A(t) = C\, \big(1 + \|\theta_0\|_\infty^2 + \|u_y\|_2^2 + (1 + \|u\|_2^2)\|v_y\|_2^2\big)$. An application of Gronwall's inequality then concludes the proof of Theorem 5.

Proof of Proposition 7

Before we provide the real proof, we would like to understand how a bound of this level can be obtained. For this purpose, we make the ansatz

$$\int_0^T \|p\|_\infty^2 \, dt < \infty.$$

Then we show that

$$\|v\|_{L^{2r}} \leq C\sqrt{r}.$$

Recall the equation for the velocity field

$$\begin{cases} u_t + uu_x + vu_y = -p_x + \nu \, u_{yy}, \\ v_t + uv_x + vv_y = -p_y + \nu \, v_{yy} + \theta \end{cases} \tag{1.16}$$

Taking the inner product of the second equation in (1.16) with $v\,|v|^{2r-2}$ and integrating by parts, we obtain

$$\frac{1}{2r}\frac{d}{dt}\int |v|^{2r} + \nu(2r-1)\int v_y^2\,|v|^{2r-2}$$
$$= (2r-1)\int p\,v_y\,|v|^{2r-2} + \int \theta\,v\,|v|^{2r-2}.$$

The last term is easy to handle and we focus on the term involving the pressure p.

$$(2r-1)\int p\,v_y\,|v|^{2r-2} = (2r-1)\int p\,|v|^{r-1}\,v_y\,|v|^{r-1}$$
$$\leq (2r-1)\,\|p\|_\infty\,\||v|^{r-1}\|_2\,\|v_y|v|^{r-1}\|_2$$
$$\leq \frac{\nu(2r-1)}{4}\,\|v_y|v|^{r-1}\|_2^2 + C\,(2r-1)\,\|p\|_\infty^2\,\|v\|_{2r-2}^{2r-2}$$
$$\leq \frac{\nu(2r-1)}{4}\,\|v_y|v|^{r-1}\|_2^2 + C\,(2r-1)\,\|p\|_\infty^2\,\|v\|_2^{\frac{2}{r-1}}\,\|v\|_{2r}^{2r-2-\frac{2}{r-1}}.$$

Then, if $\int_0^T \|p\|_\infty^2 \, dt < \infty$,

$$\frac{1}{2r}\frac{d}{dt}\|v\|_{2r}^{2r} \leq C\,(2r-1)\,\|p\|_\infty^2\,\|v\|_2^{\frac{2}{r-1}}\,\|v\|_{2r}^{2r-2-\frac{2}{r-1}}$$

would yield $\|v\|_{2r} \leq C\,\sqrt{r}$. But unfortunately, we do not know if $\int_0^T \|p\|_\infty^2 \, dt < \infty$. What we can show is the following bound.

Proposition 9

Let $(u_0, v_0, \theta_0) \in H^2(\mathbb{R}^2)$ *and let* (u, v, θ) *be the corresponding classical solution of (1.10). Then*

$$\|(u(t), v(t))\|_4^4 + \nu \int_0^t \||(u_y(\tau), v_y(\tau))|\,|(u(\tau), v(\tau))|\|_2^2\, d\tau \le M_1(t), \tag{1.17}$$

$$\|p(\cdot, t)\|_2 \le M_2(t), \qquad \int_0^t \|\nabla p(\cdot, \tau)\|_2^2\, d\tau \le M_3(t), \tag{1.18}$$

where M_1, M_2 *and* M_3 *are explicit smooth functions of* $t \in [0, \infty)$ *that depend on* ν, κ *and the initial norm* $\|(u_0, v_0, \theta_0)\|_{H^2}$.

We also need two lemmas.

Lemma 10

Let $f \in H^1(\mathbb{R}^2)$. Let $R > 0$. Denote by $B(0, R)$ the ball centered at zero with radius R and by $\chi_{B(0,R)}$ the characteristic function on $B(0, R)$. Write $f = \overline{f} + \widetilde{f}$ with

$$\overline{f} = \mathcal{F}^{-1}(\chi_{B(0,R)}\mathcal{F}f) \quad \text{and} \quad \widetilde{f} = \mathcal{F}^{-1}((1 - \chi_{B(0,R)})\mathcal{F}f). \tag{1.19}$$

Then we have the following estimates for $\overline{f}$ and $\widetilde{f}$.

(1) *There exists a pure constant C independent of f and R such that*

$$\|\overline{f}\|_{L^\infty(\mathbb{R}^2)} \leq C\sqrt{\log R}\,\|f\|_{H^1(\mathbb{R}^2)}. \tag{1.20}$$

Lemma 10

(2) For any $2 \le q < \infty$, there is a constant independent of q, R and f such that

$$\|\widetilde{f}\|_{L^q(\mathbb{R}^2)} \le C\,\frac{q}{R^{\frac{2}{q}}}\,\|\widetilde{f}\|_{H^1(\mathbb{R}^2)} \le C\,\frac{q}{R^{\frac{2}{q}}}\,\|f\|_{H^1(\mathbb{R}^2)} \tag{1.21}$$

In particular, for $q = 4$,

$$\|\widetilde{f}\|_{L^4(\mathbb{R}^2)} \le \frac{C}{\sqrt{R}}\,\|f\|_{H^1(\mathbb{R}^2)}.$$

Lemma 11

Let $q \in [2, \infty)$. Assume that $f, g, g_y, h_x \in L^2(\mathbb{R}^2)$ and $h \in L^{2(q-1)}(\mathbb{R}^2)$. Then

$$\iint_{\mathbb{R}^2} |f\, g\, h|\, dxdy \le C\, \|f\|_2\, \|g\|_2^{1-\frac{1}{q}} \|g_y\|_2^{\frac{1}{q}}\, \|h\|_{2(q-1)}^{1-\frac{1}{q}} \|h_x\|_2^{\frac{1}{q}}. \tag{1.22}$$

where C is a constant depending on q only. Two special cases of (1.22) are

$$\iint |f\, g\, h|\, dxdy \le C\, \|f\|_2\, \|g\|_2^{\frac{2}{3}}\, \|g_y\|_2^{\frac{1}{3}}\, \|h\|_4^{\frac{2}{3}}\, \|h_x\|_2^{\frac{1}{3}} \tag{1.23}$$

and

$$\iint |f\, g\, h|\, dxdy \le C\, \|f\|_2\, \|g\|_2^{\frac{1}{2}}\, \|g_y\|_2^{\frac{1}{2}}\, \|h\|_2^{\frac{1}{2}}\, \|h_x\|_2^{\frac{1}{2}}. \tag{1.24}$$

Proof of Proposition 7.

$$\begin{cases} u_t + uu_x + vu_y = -p_x + \nu\, u_{yy}, \\ v_t + uv_x + vv_y = -p_y + \nu\, v_{yy} + \theta \end{cases} \tag{1.25}$$

Taking the inner product of the second equation in (1.25) with $v\,|v|^{2r-2}$ and integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2r}\frac{d}{dt}\int |v|^{2r} + \nu(2r-1)\int v_y^2\,|v|^{2r-2} \\ &\quad = (2r-1)\int p\, v_y\, |v|^{2r-2} + \int \theta\, v\, |v|^{2r-2} \\ &\quad = (2r-1)\int \overline{p}\, v_y\, |v|^{2r-2} + (2r-1)\int \tilde{p}\, v_y\, |v|^{2r-2} + \int \theta\, v\, |v|^{2r-2}. \end{aligned}$$

By Hölder's inequality,

$$\int \theta\, v\, |v|^{2r-2} \le \|\theta\|_{2r}\, \|v\|_{2r}^{2r-1}, \tag{1.26}$$

$$\int \overline{p}\, v_y\ |v|^{2r-2} \le \|\overline{p}\|_\infty\, \|v^{r-1}\|_2\, \|v_y v^{r-1}\|_2.$$

Applying Lemma 8, we have

$$\int \tilde{p}\, v_y\ |v|^{2r-2} \le C\|\tilde{p}\|_4^{\frac{2}{3}}\, \|\tilde{p}_x\|_2^{\frac{1}{3}}\, \|v^{r-1}\|_2^{\frac{2}{3}}\, \|(r-1) v_y v^{r-2}\|_2^{\frac{1}{3}}\, \|v_y v^{r-1}\|_2.$$

Furthermore, by Hölder's inequality,

$$\left\| |v|^{r-1} \right\|_2 = \|v\|_{2(r-1)}^{r-1} \le \|v\|_2^{\frac{1}{r-1}} \|v\|_{2r}^{\frac{r(r-2)}{r-1}},$$

$$\left\| |v|^{r-2} v_y \right\|_2^2 = \int |v|^{2(r-2)} v_y^2 = \int |v|^{2(r-2)} v_y^{\frac{2(r-2)}{r-1}} v_y^{\frac{2}{r-1}} \le \|v_y\|_2^{\frac{2}{r-1}} \left\| v_y \, |v|^{r-1} \right\|_2^{\frac{2(r-2)}{r-1}}.$$

Therefore,

$$\begin{aligned} \int \overline{p}\, v_y \, |v|^{2r-2} &\le C \, \|\overline{p}\|_\infty \, \|v\|_2^{\frac{1}{r-1}} \, \|v\|_{2r}^{\frac{r(r-2)}{r-1}} \, \|v_y v^{r-1}\|_2, \\ \int \tilde{p}\, v_y \, |v|^{2r-2} &\le C \, (r-1)^{\frac{1}{3}} \, \|\tilde{p}\|_4^{\frac{2}{3}} \, \|\tilde{p}_x\|_2^{\frac{1}{3}} \, \|v\|_2^{\frac{2}{3(r-1)}} \, \|v\|_{2r}^{\frac{2r(r-2)}{3(r-1)}} \\ &\qquad \times \|v_y\|_2^{\frac{1}{3(r-1)}} \left\| v_y \, |v|^{r-1} \right\|_2^{1+\frac{(r-2)}{3(r-1)}}. \end{aligned}$$

By Young's inequality and Lemma 10,

$$(2r-1)\int \overline{p}\, v_y\ |v|^{2r-2} \le \frac{\nu}{4}(2r-1)\|v_y v^{r-1}\|_2^2 + C(2r-1)(\log R)\,\|p\|_{H^1}^2 \|v\|_2^{\frac{2}{r-1}}\ \|v\|_{2r}^{2r-2-\frac{2}{r-1}}.$$

By Young's inequality and Lemmas 10,

$$(2r-1)\int \tilde{p}\, v_y\ |v|^{2r-2} \le \frac{\nu}{4}(2r-1)\|v_y v^{r-1}\|_2^2 + C\,(2r-1)(r-1)^{\frac{2r-2}{2r-1}} \times \|\tilde{p}\|_4^{\frac{4(r-1)}{2r-1}}\ \|\tilde{p}_x\|_2^{\frac{2(r-1)}{2r-1}}\ \|v_y\|_2^{\frac{2}{2r-1}}\ \|v\|_2^{\frac{4}{2r-1}}\ \|v\|_{2r}^{2r-2-\frac{2(r+1)}{2r-1}}$$
$$\le \frac{\nu}{4}(2r-1)\|v_y v^{r-1}\|_2^2 + C\,(2r-1)(r-1)^{\frac{2r-2}{2r-1}} R^{-\frac{r-1}{2r-1}} \times \|p\|_{L^4}^{\frac{2r-2}{2r-1}}\ \|p\|_{H^1}^{\frac{4r-4}{2r-1}}\ \|v_y\|_2^{\frac{2}{2r-1}}\ \|v\|_2^{\frac{4}{2r-1}}\ \|v\|_{2r}^{2r-3-\frac{3}{2r-1}}.$$

Without loss of generality, we assume $\|v\|_{2r} \geq 1$. Inserting (1.26),(1.27) and (1.27) in (1.26), we have

$$\frac{1}{2r}\frac{d}{dt}\|v\|_{L^{2r}}^{2r} + \frac{\nu}{2}(2r-1)\int v_y^2|v|^{2r-2}\,dx$$
$$\leq C(2r-1)(\log R)\,\|p\|_{H^1}^2\|v\|_2^{\frac{2}{r-1}}\,\|v\|_{2r}^{2r-2}$$
$$+C\,(2r-1)(r-1)^{\frac{2r-2}{2r-1}}R^{-\frac{r-1}{2r-1}}\|p\|_{L^4}^{\frac{2r-2}{2r-1}}\|p\|_{H^1}^{\frac{4r-4}{2r-1}}\|v_y\|_2^{\frac{2}{2r-1}}\|v\|_2^{\frac{4}{2r-1}}\,\|v\|_{2r}^{2r-2}$$
$$+\|\theta\|_{L^{2r}}\,\|v\|_{L^{2r}}^{2r-1}.$$

Especially,

$$\begin{aligned}\frac{d}{dt}\|v\|_{L^{2r}}^2 \le C(2r-1)(\log R)\,\|p\|_{H^1}^2\|v\|_2^{\frac{2}{r-1}} \\ + C\,(2r-1)(r-1)^{\frac{2r-2}{2r-1}}R^{-\frac{r-1}{2r-1}}\|p\|_{L^4}^{\frac{2r-2}{2r-1}}\left(\|p\|_{H^1}^2+\|v_y\|_2^2\right)\|v\|_2^{\frac{4}{2r-1}} \\ + \|\theta\|_{L^{2r}}^2 + \|v\|_{L^{2r}}^2.\end{aligned}$$

Taking $R=(2r-1)^{\frac{2r-1}{2r-2}}(r-1)^2$, integrating in time and applying Propositions, we obtain

$$\|v(t)\|_{L^{2r}}^2 \le \|v_0\|_{L^{2r}}^2 + B_1(t)r\log r + B_2(t),$$

where B_1 and B_2 are explicit integrable functions. Therefore,

$$\sup_{r\ge 2}\frac{\|v(t)\|_{L^{2r}}^2}{r\log r} \le \sup_{r\ge 2}\frac{\|v_0\|_{L^{2r}}^2}{r\log r} + (B_1(t)+B_2(t)).$$

This completes the proof of Proposition 7.

This subsection proves Proposition 8.

By the Littlewood-Paley decomposition, we can write

$$f = S_{N+1} f + \sum_{j=N+1}^{\infty} \Delta_j f,$$

where Δ_j denotes the Fourier localization operator and

$$S_{N+1} = \sum_{j=-1}^{N} \Delta_j.$$

The definitions of Δ_j and S_N are now standard. Therefore,

$$\|f\|_\infty \leq \|S_{N+1} f\|_\infty + \sum_{j=N+1}^{\infty} \|\Delta_j f\|_\infty.$$

We denote the terms on the right by I and II. By Bernstein's inequality, for any $q \geq 2$,

$$|I| \leq 2^{\frac{2N}{q}} \|S_{N+1} f\|_q \leq 2^{\frac{2N}{q}} \|f\|_q.$$

Taking $q = N$, we have

$$|I| \leq 4\|f\|_N \leq 4\sqrt{N \log N} \sup_{r \geq 2} \frac{\|f\|_r}{\sqrt{r \log r}}.$$

By Bernstein's inequality again, for any $s > 1$,

$$\begin{aligned} |II| &\leq \sum_{j=N+1}^{\infty} 2^j \|\Delta_j f\|_2 = \sum_{j=N+1}^{\infty} 2^{-j(s-1)} 2^{sj} \|\Delta_j f\|_2 \\ &= C\, 2^{-(N+1)(s-1)} \|f\|_{B^s_{2,2}}. \end{aligned}$$

where C is a constant depending on s only.

By identifying $B^s_{2,2}$ with H^s, we obtain

$$\|f\|_\infty \leq 4\sqrt{N\log N}\sup_{r\geq 2}\frac{\|f\|_r}{\sqrt{r\log r}} + C\,2^{-(N+1)(s-1)}\,\|f\|_{H^s}.$$

We obtain the desired inequality (1.15) by taking

$$N = \left[\frac{1}{s-1}\log_2(e+\|f\|_{H^s})\right],$$

where $[a]$ denotes the largest integer less than or equal to a. $\square$

Our attention here focuses on the following 2D Boussinesq equations with horizontal dissipation in the vertical velocity equation and vertical dissipation in the temperature equation

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p = 0, \\ \partial_t v + u\partial_x v + v\partial_y v + \partial_y p - \partial_{xx} v = \theta, \\ \partial_t \theta + u\partial_x \theta + v\partial_y \theta - \partial_{yy}\theta = 0, \\ \partial_x u + \partial_y v = 0, \\ u(x,y,0) = u_0(x,y), \quad v(x,y,0) = v_0(x,y), \\ \theta(x,y,0) = \theta_0(x,y). \end{cases} \tag{1.27}$$

Wu, Jiahong; Xu, Xiaojing; Ye, Zhuan, J.Math. Fluid Mech.(2017) established several global bounds, which may be useful in the eventual resolution of whether or not (1.27) is globally well-posed.

Theorem 12 (Wu, Jiahong; Xu, Xiaojing; Ye, Zhuan (2017))

Assume that $(\vec{u}_0, \theta_0) \in H^\sigma\left(\mathbb{R}^2\right)$ *with* $\sigma > 2$ *and* $\nabla \cdot \vec{u}_0 = 0$*. Let* $(\vec{u}, \theta)$ *be the corresponding solution of (1.27). Then,* $(\vec{u}, \theta)$ *admits the following global bounds, for any* $T > 0$ *and* $t \leq T$*,*

$$\|\vec{u}(t)\|_{H^1}^2 + \int_0^t \|\partial_x \nabla v(\tau)\|_{L^2}^2 \, d\tau \leq C,$$

where $C = C\left(T, \vec{u}_0, \theta_0\right)$*;*

$$\|\theta(t)\|_{H^1}^2 + \int_0^t \|\partial_y \nabla \theta(\tau)\|_{L^2}^2 \, d\tau \leq C,$$

where $C = C\left(T, \vec{u}_0, \theta_0\right)$*;*

Theorem 12 (Wu, Jiahong; Xu, Xiaojing; Ye, Zhuan (2017))

$$\left\| |\partial_y|^{1+s}\,\theta(t)\right\|_{L^2}^2 + \int_0^t \left\| |\partial_y|^{2+s}\,\theta(\tau)\right\|_{L^2}^2 d\tau \le C,$$

$$\int_0^t \|\partial_y \theta(\tau)\|_{L^\infty}^2 \, d\tau \le C,$$

where $0 < s < \frac{1}{2}$ and $C = C\left(T, \vec{u}_0, \theta_0\right)$;

$$\|\partial_x \theta(t)\|_{L^q} \le C, \quad 2 \le q < \infty$$

where $C = C\left(T, q, \vec{u}_0, \theta_0\right)$.

The anisotropic Sobolev inequalities:

Lemma 13

The following anisotropic Sobolev inequalities hold, $\frac{1}{2} < \gamma \leq 1$,

$$\int_{\mathbb{R}^2} |fgh| dxdy \leq C \|f\|_{L^2} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_x g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}^{\frac{1}{2}} \|\partial_y h\|_{L^2}^{\frac{1}{2}},$$

$$\int_{\mathbb{R}^2} |fgh| dxdy \leq C \|f\|_{L^2} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_x g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}^{\frac{2\gamma-1}{2\gamma}} \||\partial_y|^{\gamma} h\|_{L^2}^{\frac{1}{2\gamma}}.$$

$$\begin{cases} \partial_t u + (u,\ v)\cdot\nabla u + \partial_x P = \mu_1\partial_{xx}u + \mu_2\partial_{yy}u, \\ \partial_t v + (u,\ v)\cdot\nabla v + \partial_y P = \nu_1\partial_{xx}v + \nu_2\partial_{yy}v + \theta, \\ \partial_t\theta + (u,\ v)\cdot\nabla\theta = \eta_1\partial_{xx}\theta + \eta_2\partial_{yy}\theta, \\ \partial_x u = -\partial_y v, \\ (u,\ v)(x,y,0) = (u_0,\ v_0)(x,y), \quad \theta(x,y,0) = \theta_0(x,y). \end{cases} \tag{1.28}$$

Global results to the above system:

$$(I) \quad \mu_1 = \nu_1 > 0, \quad \mu_2 = \nu_2 = \eta_1 = \eta_2 = 0.$$

$Danchin - Paicu, 2011, M3AS;$

$Larios - Lunasin - Titi, 2013, JDE.(weaker\ space)$

$$(II) \quad \mu_1 = \nu_2 = \eta_1 = \eta_2 = 0, \quad \mu_2 > 0,\ \nu_1 > 0,$$
$$or\, \mu_1 = \mu_2 = \eta_1 = \eta_2 = 0, \quad \nu_1 > 0,\ \nu_2 > 0.$$

$Adhikari - Cao - Shang - Wu - Xu - Ye, 2016, JDE;$

$$(III) \quad \mu_1 = \nu_1 = 0, \quad \mu_2 = \nu_2 > 0, \quad \eta_1 = 0, \quad \eta_2 > 0.$$

$Cao - Wu, 2013, ARMA(H^2); Li - Titi, 2016, ARMA(L^2_x).$

Our aim: between $(-\Delta)^\alpha$ and $(-\partial_{x_1x_1})$

Boussinesq equations with fractional-anisotropic dissipation:

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \mu\Lambda_{x_1}^{2\alpha}u + \nabla p = \theta e_2, \\ \partial_t\theta + (u\cdot\nabla)\theta + \nu\Lambda_{x_1}^{2\beta}\theta = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.29}$$

where $\Lambda_{x_1}^\gamma = (-\partial_{x_1x_1})^{\gamma/2}$ is the pseudodifferential operator defined *via* the Fourier transform

$$\widehat{\Lambda_{x_1}^\gamma u}(\xi) = |\xi_1|^\gamma \widehat{u}(\xi).$$

Vorticity equation:

$$\partial_t\omega + (u\cdot\nabla)\omega + \mu\Lambda_{x_1}^{2\alpha}\omega = \partial_{x_1}\theta.$$

$$\|\omega(t)\|_{L^\infty} \le \|\partial_{x_1}\theta(t)\|_{L^\infty}.$$

Motivation

Blow-up criterion:

$$\int_0^T \|\nabla u(\cdot,\, t)\|_{L^\infty}\, dt < \infty \quad \Longleftrightarrow \quad Global \ \ regularity.$$

One Way: By the interpolation inequality and Biot-Savart Law:

$$\begin{aligned} \|\nabla u\|_{L^\infty} &\leq C(\|\nabla u\|_{L^2} + \|\Lambda_{x_1}^{1+\alpha}\nabla u\|_{L^2} + \|\Lambda_{x_2}\nabla u\|_{L^2}) \\ &\leq C(\|\omega\|_{L^2} + \|\Lambda_{x_1}^{\alpha}\nabla\omega\|_{L^2} + \|\nabla\omega\|_{L^2}). \end{aligned}$$

We need estimate:

$$\int_0^T \|\omega(t)\|_{L^2} + \|\Lambda_{x_1}^{\alpha}\nabla\omega(t)\|_{L^2} + \|\nabla\omega(t)\|_{L^2}\, dt \leq C.$$

Motivation

Blow-up criterion:

$$\int_0^T \|\nabla u(\cdot,\, t)\|_{L^\infty}\, dt < \infty \quad \Longleftrightarrow \quad Global \ \ regularity.$$

Another way: By logarithmic Sobolev interpolation inequality:

$$\|\nabla u\|_{L^\infty} \le C + C\left(\sup_{q\ge 2} \frac{\|\nabla u\|_{L^q}}{q}\right) \ln\left(e + \|\omega\|_{H^\sigma}\right),\ \forall \sigma > 1. (1.30)$$

By the Biot-Savart law:

$$\sup_{q\ge 2} \frac{\|\nabla u(t)\|_{L^q}}{q} \le C\|\omega(t)\|_{L^2\cap L^\infty} \quad \left(\|\nabla u(t)\|_{L^q} \le C\frac{q^2}{q-1}\|\omega(t)\|_{L^q}\right).$$

$$\|\omega(t)\|_{L^\infty} \le \|\partial_{x_1}\theta\|_{L^\infty} \le C(\|\nabla\theta\|_{L^2} + \|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2} + \|\Lambda_{x_1}^{2\beta+\frac{1}{2}}\theta\|_{L^2}).$$

Main Result

Our main Theorem is as follows:

Theorem 6 (J. Wu-Xu-Z. Ye, 2018, JMPA)

If α and β are in any one of the two ranges:

$$(1)\quad \alpha+\beta>1, \quad \frac{1}{2}<\alpha\le 1, \qquad \frac{1}{2}\ge\beta>\beta_0,$$

$$(2)\quad \alpha+\beta\ge 1, \qquad \beta>\frac{2+\sqrt{2}}{4},$$

where

$$\beta_0=\frac{2\alpha}{\sqrt{16\alpha^4-16\alpha^3+28\alpha^2-12\alpha+1}+4\alpha^2-2\alpha+1}, \tag{1.31}$$

then (1.29) admits a unique global solution.

The Sketch of Proof: The basic estimates

Lemma 7

Assume (u_0, θ_0) satisfies the assumptions stated in Theorem 6. Then the corresponding solution (u, θ) of (1.29) admits the following bounds, for any $t > 0$,

$$\|\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda_{x_1}^{\beta}\theta(\tau)\|_{L^2}^2 \, d\tau \le \|\theta_0\|_{L^2}^2, \tag{1.32}$$

$$\|u(t)\|_{L^2}^2 + \int_0^t \|\Lambda_{x_1}^{\alpha} u(\tau)\|_{L^2}^2 \, d\tau \le (\|u_0\|_{L^2} + t\|\theta_0\|_{L^2})^2, \tag{1.33}$$

$$\|\theta(t)\|_{L^\infty} \le \|\theta_0\|_{L^\infty}, \tag{1.34}$$

$$\|\theta(t)\|_{L^p}^p + C\int_0^t \int_{\mathbb{R}} \|\theta(., x_2)\|^p_{\dot{\mathcal{B}}^{\frac{2\beta}{p}}_{p,p}} \, dx_2 d\tau \le \|\theta_0\|_{L^p}^p, \tag{1.35}$$

where $\dot{\mathcal{B}}^s_{p,r}$ denotes the homogeneous Besov space associated with the variable x_1.

The Sketch of Proof: The basic estimates (Remark)

In order to derive (1.35), the following lower bound involving the fractional dissipation is also needed (see Chamorro and Lemarié-Rieusset's book), for any $\beta \in (0,1)$,

$$
\begin{aligned}
& \int_{\mathbb{R}} \int_{\mathbb{R}} \Lambda_{x_1}^{2\beta} \theta(x_1, x_2) \, |\theta(x_1, x_2)|^{p-2} \theta(x_1, x_2) \, dx_1 \, dx_2 \\
\geq \quad & C \int_{\mathbb{R}} \|\theta(., x_2)\|^p_{\dot{\mathcal{B}}^{\frac{2\beta}{p}}_{p,p}} \, dx_2.
\end{aligned}
$$

The Sketch of Proof: The global H^1-bound for u

Based on $\alpha + \beta \geq 1$, we first have

Proposition 1

Let $\alpha + \beta \geq 1$. Assume (u_0, θ_0) satisfies the assumptions stated in Theorem 6 and let (u, θ) be the corresponding solution. Then, for any $t > 0$,

$$\|\nabla u(t)\|_{L^2}^2 + \int_0^t \|\Lambda_{x_1}^{\alpha} \nabla u(\tau)\|_{L^2}^2 \, d\tau \leq C(t, u_0, \theta_0), \qquad (1.36)$$

where $C(t, u_0, \theta_0)$ is a constant depending on t and the initial data (u_0, θ_0).

The Sketch of Proof: The global L^p bound for ω

When $\alpha + \beta > 1$, we also obtain a global L^p bound for ω, which plays an important role in the proof of a global H^1 bound for θ in the case when $\beta \le \frac{1}{2}$.

Proposition 2

Let $\alpha + \beta > 1$. Assume (u_0, θ_0) satisfies the assumptions stated in Theorem 6 and let (u, θ) be the corresponding solution. Then, for any $2 \le p < 2(\alpha + \beta)$ and for any $t > 0$,

$$\|\omega(t)\|_{L^p}^p + \int_0^t \int_{\mathbb{R}} \|\omega(.,x_2)\|^p_{\dot{\mathcal{B}}^{\frac{2\alpha}{p}}_{p,p}} \, dx_2 d\tau \le C(t,\, u_0,\, \theta_0), \quad (1.37)$$

where $C(t, u_0, \theta_0)$ is a constant depending on t and the initial data (u_0, θ_0).

The Sketch of Proof: The global L^p bound for ω (continue)

To prove this global bound (1.37), we make use of the upper bound of the form

$$\left\|\Lambda^{\kappa}_{x_1}(|\omega|^{p-2}\omega)(.,x_2)\right\|_{L^{\frac{p}{p-1}}_{x_1}} \le C\|\omega(.,x_2)\|_{\mathcal{B}^{\sigma}_{p,\frac{p}{p-1}}}\left\||\omega(.,x_2)|^{p-2}\right\|_{L^{\frac{p}{p-2}}_{x_1}},$$

where $\sigma > \kappa$. Here $\mathcal{B}^s_{p,r}$ denotes an inhomogeneous Besov space associated with the variable x_1, whose norm is given by

$$\|f\|_{B^s_{p,q}} \approx \|f\|_{\dot{\mathcal{B}}^s_{p,q}} + \|f\|_{L^p}.$$

$$\|f\|_{\dot{\mathcal{B}}^s_{p,q}} = \left(\sum_{q\in\mathbb{Z}} 2^{jqs}\|\dot{\Delta}_j f\|^r_{L^p}\right)^{\frac{1}{q}}, \quad \forall r < \infty$$

$$\approx \left[\int_{\mathbb{R}^d} \frac{\|f(x+\cdot)-f(\cdot)\|^q_{L^p(\mathbb{R}^d)}}{|x|^{d+sq}}\,dx\right]^{\frac{1}{q}}$$

The Sketch of Proof: The global H^1 for θ

In the case when $\beta > \frac{1}{2}$, the global bound holds for any $\alpha + \beta \geq 1$.

Proposition 3

Let $\beta > \frac{1}{2}$ and $\alpha + \beta \geq 1$. Assume (u_0, θ_0) satisfies the assumptions stated in Theorem 6 and let (u, θ) be the corresponding solution. Then, for any $t > 0$,

$$\|\nabla\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda_{x_1}^{\beta}\nabla\theta(\tau)\|_{L^2}^2\, d\tau \leq C(t,\, u_0,\, \theta_0), \tag{1.38}$$

where $C(t,\, u_0,\, \theta_0)$ is a constant depending on t and the initial data (u_0, θ_0).

The Sketch of Proof: The global H^1 for θ

In the complement case when $\beta \leq \frac{1}{2}$, the global bound is given by the following proposition.

Proposition 4

Assume α and β satisfy

$$\frac{1}{2} < \alpha \leq 1, \qquad \frac{1}{2} \geq \beta > \beta_0, \qquad \alpha + \beta > 1. \tag{1.39}$$

Assume (u_0, θ_0) satisfies the assumptions stated in Theorem 6 and let (u, θ) be the corresponding solution. Then, for any $t > 0$,

$$\|\nabla\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda_{x_1}^{\beta} \nabla\theta(\tau)\|_{L^2}^2 \, d\tau \leq C(t, u_0, \theta_0), \tag{1.40}$$

where $C(t, u_0, \theta_0)$ is a constant depending on t and the initial data (u_0, θ_0).

The Sketch of Proof

In order to prove the global H^1 for θ above, we need anisotropic Sobolev inequalities.

Lemma 8 (anisotropic Sobolev inequality)

Assume $p,\ q \in [2,\ \infty]$, *and* $\gamma_1 > \frac{1}{p}$ *and* $\gamma_2 > \frac{1}{q}$. *Then, for a constant* $C = C(p, q, \gamma_1, \gamma_2) > 0$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f\, g\, h|\, dx_1 dx_2 \quad \leq \quad C\, \|f\|_{L^q_{x_2} L^p_{x_1}}\, \|g\|_{L^2}^{1-\frac{1}{\gamma_1 p}}\, \|\Lambda^{\gamma_1}_{x_1} g\|_{L^2}^{\frac{1}{\gamma_1 p}} \times \|h\|_{L^2}^{1-\frac{1}{\gamma_2 q}}\, \|\Lambda^{\gamma_2}_{x_2} h\|_{L^2}^{\frac{1}{\gamma_2 q}}. \qquad (1.41)$$

The Sketch of Proof

We start with the one-dimensional Sobolev inequality

$$\|g\|_{L_{x_1}^{\frac{2p}{p-2}}(\mathbb{R})} \le C\|g\|_{L_{x_1}^2(\mathbb{R})}^{1-\frac{1}{\gamma_1 p}} \|\Lambda_{x_1}^{\gamma_1} g\|_{L_{x_1}^2(\mathbb{R})}^{\frac{1}{\gamma_1 p}}, \; p \in [2,\infty], \; \gamma_1 > \frac{1}{p}.$$

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} |f\,g\,h|\,dx_1 dx_2 &\le C \int_{\mathbb{R}} \|f\|_{L_{x_1}^p} \|g\|_{L_{x_1}^{\frac{2p}{p-2}}} \|h\|_{L_{x_1}^2}\,dx_2 \\
&\le C \int_{\mathbb{R}} \|f\|_{L_{x_1}^p} \|g\|_{L_{x_1}^2}^{1-\frac{1}{\gamma_1 p}} \|\Lambda_{x_1}^{\gamma_1} g\|_{L_{x_1}^2}^{\frac{1}{\gamma_1 p}} \|h\|_{L_{x_1}^2}\,dx_2 \\
&\le C\Big(\int_{\mathbb{R}} \|f\|_{L_{x_1}^p}^q\,dx_2\Big)^{\frac{1}{q}} \Big(\int_{\mathbb{R}} \|g\|_{L_{x_1}^2}^2\,dx_2\Big)^{\frac{\gamma_1 p-1}{2\gamma_1 p}} \qquad (1.42) \\
&\quad \times \Big(\int_{\mathbb{R}} \|\Lambda_{x_1}^{\gamma_1} g\|_{L_{x_1}^2}^2\,dx_2\Big)^{\frac{1}{2\gamma_1 p}} \|h\|_{L_{x_2}^{\frac{2q}{q-2}} L_{x_1}^2} \\
&= C\|f\|_{L_{x_2}^q L_{x_1}^p} \|g\|_{L^2}^{1-\frac{1}{\gamma_1 p}} \|\Lambda_{x_1}^{\gamma_1} g\|_{L^2}^{\frac{1}{\gamma_1 p}} \|h\|_{L_{x_2}^{\frac{2q}{q-2}} L_{x_1}^2}. \qquad (1.43)
\end{aligned}$$

The Sketch of Proof

$$\begin{aligned}
&\|h\|_{L_{x_2}^{\frac{2q}{q-2}} L_{x_1}^2} &(1.44)\\
\leq\ & C\Big(\int_{\mathbb{R}} \|h(x_1,.)\|^2_{L_{x_2}^{\frac{2q}{q-2}}}\, dx_1\Big)^{\frac{1}{2}}\\
\leq\ & C\Big(\int_{\mathbb{R}} \|h(x_1,.)\|_{L^2_{x_2}}^{2-\frac{2}{\gamma_2 q}} \|\Lambda_{x_2}^{\gamma_2} h(x_1,.)\|_{L^2_{x_2}}^{\frac{2}{\gamma_2 q}}\, dx_1\Big)^{\frac{1}{2}}\\
\leq\ & C\Big(\int_{\mathbb{R}} \|h(x_1,.)\|^2_{L^2_{x_2}}\, dx_1\Big)^{\frac{\gamma_2 q-1}{2\gamma_2 q}} \Big(\int_{\mathbb{R}} \|\Lambda_{x_2}^{\gamma_2} h(x_1,.)\|^2_{L^2_{x_2}}\, dx_1\Big)^{\frac{1}{2\gamma_2 q}}\\
=\ & C\|h\|_{L^2}^{1-\frac{1}{\gamma_2 q}} \|\Lambda_{x_2}^{\gamma_2} h\|_{L^2}^{\frac{1}{\gamma_2 q}}. &(1.45)
\end{aligned}$$

The Sketch Proof

Lemma 9 (anisotropic Sobolev inequality)

Let $p, q \in [2, \infty]$. Then,

$$\begin{aligned} \|f\|_{L^q_{x_2} L^p_{x_1}} \leq \; & C \|f\|_{L^2_{x_1 x_2}}^{\gamma_1 \gamma_2} \|\Lambda^{\sigma_2}_{x_2} f\|_{L^2_{x_1 x_2}}^{\gamma_1 (1-\gamma_2)} \|\Lambda^{\sigma_1}_{x_1} f\|_{L^2_{x_1 x_2}}^{(1-\gamma_1)\gamma_2} \\ & \times \|\Lambda^{\sigma_2}_{x_2} \Lambda^{\sigma_1}_{x_1} f\|_{L^2_{x_1 x_2}}^{(1-\gamma_1)(1-\gamma_2)}, \end{aligned} \tag{1.46}$$

$\gamma_1, \gamma_2 \in [0, 1]$, $\sigma_1 \geq 0$ *and* $\sigma_2 \geq 0$ *satisfy*

$$(1-\gamma_1)\,\sigma_1 = \frac{1}{2} - \frac{1}{p}, \qquad (1-\gamma_2)\,\sigma_2 = \frac{1}{2} - \frac{1}{q}.$$

The Sketch Proof

Lemma 10 (anisotropic Sobolev inequality (continue))

$$\|f\|_{L^q_{x_2}L^p_{x_1}} \le C\,\|f\|^{\rho}_{L^2_{x_1x_2}}\,\|\Lambda^{\sigma_1+\sigma_2}_{x_2}f\|^{\frac{\sigma_2}{\sigma_1+\sigma_2}\gamma_1(1-\gamma_2)}_{L^2_{x_1x_2}}$$
$$\times\,\|\Lambda^{\sigma_1+\sigma_2}_{x_1}f\|^{\frac{\sigma_1}{\sigma_1+\sigma_2}(1-\gamma_1)\gamma_2}_{L^2_{x_1x_2}}\,\|\Lambda^{\sigma_2}_{x_2}\Lambda^{\sigma_1}_{x_1}f\|^{(1-\gamma_1)(1-\gamma_2)}_{L^2_{x_1x_2}},$$

where ρ is given by

$$\rho=\gamma_1\gamma_2+\frac{\sigma_1}{\sigma_1+\sigma_2}\gamma_1(1-\gamma_2)+\frac{\sigma_2}{\sigma_1+\sigma_2}(1-\gamma_1)\gamma_2.$$

The Sketch Proof

Lemma 11 (anisotropic Sobolev inequality (continue))

In the special case when $p = q = \infty$, (1.46) becomes

$$\begin{aligned}\|f\|_{L^\infty} \leq{}& C\|f\|_{L^2}^{(1-\frac{1}{2\sigma_1})(1-\frac{1}{2\sigma_2})}\|\Lambda_{x_2}^{\sigma_2}f\|_{L^2}^{(1-\frac{1}{2\sigma_1})\frac{1}{2\sigma_2}}\|\Lambda_{x_1}^{\sigma_1}f\|_{L^2}^{\frac{1}{2\sigma_1}(1-\frac{1}{2\sigma_2})}\\ &\times\|\Lambda_{x_1}^{\sigma_1}\Lambda_{x_2}^{\sigma_2}f\|_{L^2}^{\frac{1}{4\sigma_1\sigma_2}}, \qquad (1.47)\end{aligned}$$

where $\sigma_1, \sigma_2 > \frac{1}{2}$.

The Sketch Proof: $\alpha > \frac{1}{2}$, $\beta > \beta_0$

Key bound:

Proposition 5 (H^1 bound of ω for the case $\alpha > \frac{1}{2}$ and $\beta > \beta_0$)

The $\omega = \nabla \times u$ satisfies

$$\|\nabla\omega(t)\|_{L^2}^2 + \int_0^t \|\Lambda_{x_1}^{\alpha}\nabla\omega(\tau)\|_{L^2}^2\, d\tau \le C(t, u_0, \theta_0), \tag{1.48}$$

where $C(t, u_0, \theta_0)$ is a constant depending on t and the initial data (u_0, θ_0).

This key estimate allows us to show higher regularity below.

To prove Proposition 5, we first derive a global H^2-bound for u. Taking the gradient of the vorticity equation

$$\partial_t \omega + (u \cdot \nabla)\omega + \Lambda_{x_1}^{2\alpha} \omega = \partial_{x_1} \theta,$$

and dotting it with $\nabla\omega$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla\omega(t)\|_{L^2}^2 + \left\|\Lambda_{x_1}^{\alpha} \nabla\omega\right\|_{L^2}^2 &= -\int_{\mathbb{R}^2} (\nabla u \cdot \nabla\omega) \cdot \nabla\omega dx - \int_{\mathbb{R}^2} \partial_{x_1} \theta \Delta\omega dx \\ &:= N_1 + N_2. \end{aligned}$$

where N_1 is a quadratic form and can be explicitly written as

$$\begin{aligned}N_1 =& -\int_{\mathbb{R}^2} \partial_{x_1} u_1 \partial_{x_1}\omega \partial_{x_1}\omega dx - \int_{\mathbb{R}^2} \partial_{x_1} u_2 \partial_{x_2}\omega \partial_{x_1}\omega dx \\ & - \int_{\mathbb{R}^2} \partial_{x_2} u_1 \partial_{x_1}\omega \partial_{x_2}\omega dx - \int_{\mathbb{R}^2} \partial_{x_2} u_2 \partial_{x_2}\omega \partial_{x_2}\omega dx \\ :=& N_{11} + N_{12} + N_{13} + N_{14}.\end{aligned}$$

Applying Lemma 8 with $p = q = 2$, $\gamma_1 = \gamma_2 = \alpha > \frac{1}{2}$ and Hölder's inequality, we obtain

$$\begin{aligned}
N_{11}, N_{12} &\leq C\|\nabla\omega\|_{L^2} \|\partial_{x_1} u\|_{L^2}^{1-\frac{1}{2\alpha}} \left\|\Lambda_{x_2}^{\alpha}\partial_{x_1} u\right\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_{x_1}\omega\|_{L^2}^{1-\frac{1}{2\alpha}} \left\|\Lambda_{x_1}^{\alpha}\partial_{x_1}\omega\right\|_{L^2}^{\frac{1}{2\alpha}} \\
&\leq C\|\nabla\omega\|_{L^2}^{2-\frac{1}{2\alpha}} \|\omega\|_{L^2}^{1-\frac{1}{2\alpha}} \left\|\Lambda_{x_2}^{\alpha}\partial_{x_1} u\right\|_{L^2}^{\frac{1}{2\alpha}} \left\|\Lambda_{x_1}^{\alpha}\nabla\omega\right\|_{L^2}^{\frac{1}{2\alpha}} \\
&\leq C\|\nabla\omega\|_{L^2}^{2-\frac{1}{2\alpha}} \|\omega\|_{L^2}^{1-\frac{1}{2\alpha}} \left\|\Lambda_{x_1}^{\alpha}\nabla u\right\|_{L^2}^{\frac{1}{2\alpha}} \left\|\Lambda_{x_1}^{\alpha}\nabla\omega\right\|_{L^2}^{\frac{1}{2\alpha}} \\
&\leq \frac{1}{16}\left\|\Lambda_{x_1}^{\alpha}\nabla\omega\right\|_{L^2}^{2} + C\|\omega\|_{L^2}^{\frac{4\alpha-2}{4\alpha-1}} \left\|\Lambda_{x_1}^{\alpha}\nabla u\right\|_{L^2}^{\frac{2}{4\alpha-1}} \|\nabla\omega\|_{L^2}^{2}
\end{aligned}$$

where in the third line we have applied the following estimate

$$\left\|\Lambda_{x_2}^{\alpha}\partial_{x_1} u\right\|_{L^2} = \left\||\xi_2|^{\alpha}|\xi_1|\,\widehat{u}(\xi)\right\|_{L^2} \leq \left\||\xi_1|^{\alpha}|\xi|\widehat{u}(\xi)\right\|_{L^2} = \left\|\Lambda_{x_1}^{\alpha}\nabla u\right\|_{L^2}. \tag{1.49}$$

Once again, Lemma 8 and the inequalities (1.49) entail

$$\begin{aligned}
N_{13} &\le C\,\|\partial_{x_2}\omega\|_{L^2}\,\|\partial_{x_2}u_1\|_{L^2}^{1-\frac{1}{2\alpha}}\,\left\|\Lambda_{x_1}^{\alpha}\partial_{x_2}u_1\right\|_{L^2}^{\frac{1}{2\alpha}}\,\|\partial_{x_1}\omega\|_{L^2}^{1-\frac{1}{2\alpha}}\,\left\|\Lambda_{x_2}^{\alpha}\partial_{x_1}\omega\right\|_{L^2}^{\frac{1}{2\alpha}} \\
&\le C\|\nabla\omega\|_{L^2}^{2-\frac{1}{2\alpha}}\|\omega\|_{L^2}^{1-\frac{1}{2\alpha}}\left\|\Lambda_{x_1}^{\alpha}\partial_{x_2}u\right\|_{L^2}^{\frac{1}{2\alpha}}\left\|\Lambda_{x_2}^{\alpha}\partial_{x_1}\omega\right\|_{L^2}^{\frac{1}{2\alpha}} \\
&\le C\|\nabla\omega\|_{L^2}^{2-\frac{1}{2\alpha}}\|\omega\|_{L^2}^{1-\frac{1}{2\alpha}}\left\|\Lambda_{x_1}^{\alpha}\nabla u\right\|_{L^2}^{\frac{1}{2\alpha}}\left\|\Lambda_{x_1}^{\alpha}\nabla\omega\right\|_{L^2}^{2\alpha} \\
&\le \frac{1}{16}\left\|\Lambda_{x_1}^{\alpha}\nabla\omega\right\|_{L^2}^{2} + C\|\omega\|_{L^2}^{\frac{4\alpha-2}{2\alpha-1}}\left\|\Lambda_{x_1}^{\alpha}\nabla u\right\|_{L^2}^{\frac{2}{4\alpha-1}}\|\nabla\omega\|_{L^2}^{2}.
\end{aligned}$$

$$
\begin{aligned}
N_{14} &= -\int_{\mathbb{R}^2} \partial_{x_2} u_1 \partial_{x_2}\omega \partial_{x_2}\omega dx \\
&\leq C \|\partial_{x_2}\omega\|_{L^2} \|\partial_{x_1} u_1\|_{L^2}^{1-\frac{1}{2\alpha}} \left\|\Lambda_{x_2}^{\alpha} \partial_{x_1} u_1\right\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_{x_2}\omega\|_{L^2}^{1-\frac{1}{2\alpha}} \left\|\Lambda_{x_1}^{\alpha} \partial_{x_2}\omega\right\|_{L^2}^{\frac{1}{2\alpha}} \\
&\leq C\|\nabla\omega\|_{L^2}^{2-\frac{1}{2\alpha}} \|\omega\|_{L^2}^{1-\frac{1}{2\alpha}} \left\|\Lambda_{x_1}^{\alpha} \nabla u\right\|_{L^2}^{\frac{1}{2\alpha}} \left\|\Lambda_{x_1}^{\alpha} \nabla\omega\right\|_{L^2}^{\frac{1}{2\alpha}} \\
&\leq \frac{1}{16} \left\|\Lambda_{x_1}^{\alpha} \nabla\omega\right\|_{L^2}^{2} + C\|\omega\|_{L^2}^{\frac{4\alpha-2}{4\alpha-1}} \left\|\Lambda_{x_1}^{\alpha} \nabla u\right\|_{L^2}^{\frac{2}{4\alpha-1}} \|\nabla\omega\|_{L^2}^{2}.
\end{aligned}
$$

Combining these estimates, we obtain

$$
\begin{aligned}
N_1 &= -\int_{\mathbb{R}^2} (\nabla u \cdot \nabla\omega) \cdot \nabla\omega dx \\
&\leq \frac{1}{4} \left\|\Lambda_{x_1}^{\alpha} \nabla\omega\right\|_{L^2}^{2} + C\|\omega\|_{L^2}^{\frac{4\alpha-2}{4\alpha-1}} \left\|\Lambda_{x_1}^{\alpha} \nabla u\right\|_{L^2}^{\frac{2}{4\alpha-1}} \|\nabla\omega\|_{L^2}^{2}.
\end{aligned}
$$

By Hölder's inequality and the interpolation inequality

$$
\begin{aligned}
|N_2| &\leq C\left\|\Lambda_{x_1}^{\alpha}\nabla\omega\right\|_{L^2}\left\|\Lambda_{x_1}^{1-\alpha}\nabla\theta\right\|_{L^2}\\
&\leq C\left\|\Lambda_{x_1}^{\alpha}\nabla\omega\right\|_{L^2}\left\|\Lambda_{x_1}^{\beta}\nabla\theta\right\|_{L^2}^{\frac{1-\alpha}{\beta}}\|\nabla\theta\|_{L^2}^{\frac{\alpha+\beta-1}{\beta}}\\
&\leq \frac{1}{16}\|\Lambda_{x_1}^{\alpha}\nabla\omega\|_{L^2}^2 + C(\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2)
\end{aligned}
$$

Combining the estimates above and invoking the fact $\frac{2}{4\alpha-1} \leq 2$, we have

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla\omega(t)\|_{L^2}^2 + \|\Lambda_{x_1}^{\alpha}\nabla\omega\|_{L^2}^2\\
\leq& C\|\omega\|_{L^2}^{\frac{4\alpha-2}{4\alpha-1}}\|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}^{\frac{2}{4\alpha-1}}\|\nabla\omega\|_{L^2}^2 + C(\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2)\\
\leq& C\|\omega\|_{L^2}^{\frac{4\alpha-2}{4\alpha-1}}(1+\|\Lambda_{x_1}^{\alpha}\nabla u\|_{L^2}^2)\|\nabla\omega\|_{L^2}^2 + C(\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2).
\end{aligned}
$$

Applying the Gronwall inequality lead to

$$\|\nabla\omega(t)\|_{L^2}^2 + \int_0^t \left\|\Lambda_{x_1}^{\alpha}\nabla\omega(\tau)\right\|_{L^2}^2 d\tau \leq C\left(t, u_0, \theta_0\right). \tag{1.50}$$

This we get a global H^2-bound for u.
Now we are ready to establish the global H^s-estimate.

It follows from energy estimates that

$$\begin{aligned}
&\frac{d}{dt}(\|u(t)\|_{H^s}^2 + \|\theta(t)\|_{H^s}^2) + \|\Lambda_{x_1}^{\alpha} u\|_{H^s}^2 + \|\Lambda_{x_1}^{\beta}\theta\|_{H^s}^2 \\
\leq & C(1 + \|\nabla u\|_{L^\infty} + \|\nabla\theta\|_{L^\infty})(\|u\|_{H^s}^2 + \|\theta\|_{H^s}^2)
\end{aligned}$$

To bound $\|\nabla u\|_{L^\infty}$, we recall the interpolation inequality

$$\|h\|_{L^\infty} \leq C(\|h\|_{L^2} + \|\Lambda_{x_1}^{\delta_1} h\|_{L^2} + \|\Lambda_{x_2}^{\delta_2} h\|_{L^2}), \quad \frac{1}{\delta_1} + \frac{1}{\delta_2} < 2, \tag{1.51}$$

which implies that

$$\begin{aligned}
\|\nabla u\|_{L^\infty} &\leq C\left(\|\nabla u\|_{L^2} + \left\|\Lambda_{x_1}^{1+\alpha}\nabla u\right\|_{L^2} + \|\Lambda_{x_2}\nabla u\|_{L^2}\right) \\
&\leq C\left(\|\omega\|_{L^2} + \left\|\Lambda_{x_1}^{\alpha}\nabla\omega\right\|_{L^2} + \|\nabla\omega\|_{L^2}\right)
\end{aligned}$$

The global H^2-bound in (1.50) then yields

$$\int_0^t \|\nabla u(s)\|_{L^\infty}^2 ds < \infty$$

This bound leads to a global bound for $\|\nabla\theta\|_{L^\infty}$. In fact, for any $q \in [1, \infty]$,

$$\|\nabla\theta\|_{L^q} \leq \|\nabla\theta_0\|_{L^q} \exp\left[\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau\right]$$

It is then clear that

$$\|u(t)\|_{H^s}^2 + \|\theta(t)\|_{H^s}^2 + \int_0^t \left(\left\|\Lambda_{x_1}^{\alpha} u(\tau)\right\|_{H^s}^2 + \left\|\Lambda_{x_1}^{\beta} \theta(\tau)\right\|_{H^s}^2\right) d\tau < \infty$$

This obtain Proposition 6. $\square$

The Sketch Proof: $\alpha > \frac{1}{2}$, $\beta > \beta_0$

Proposition 6 (Higher regularity: $\alpha > \frac{1}{2}$ and $\beta > \beta_0$)

Assume that $(u_0, \theta_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ *with* $s > 2$ *and* $\nabla \cdot u_0 = 0$. *Assume* α *and* β *satisfy*

$$\frac{1}{2} < \alpha \le 1, \qquad \frac{1}{2} \ge \beta > \beta_0, \qquad \alpha + \beta > 1.$$

then the corresponding solution of (1.29) obeys the following global bound

$$\|u(t)\|_{H^s}^2 + \|\theta(t)\|_{H^s}^2 + \int_0^t \left(\|\Lambda_{x_1}^{\alpha} u(\tau)\|_{H^s}^2 + \|\Lambda_{x_1}^{\beta} \theta(\tau)\|_{H^s}^2 \right) d\tau < \infty.$$

The Sketch Proof: $\alpha+\beta\geq 1,\ \beta>\frac{2+\sqrt{2}}{4}$

Proposition 7

Assume (u_0,θ_0) satisfies the assumptions stated in Theorem 6. If $\alpha+\beta\geq 1$ and $\beta>\frac{1}{2}$, then there holds

$$\|\Lambda_{x_1}^{\beta+\frac{1}{2}}\theta(t)\|_{L^2}^2+\int_0^t\|\Lambda_{x_1}^{2\beta+\frac{1}{2}}\theta(s)\|_{L^2}^2\,ds\leq C(t,u_0,\theta_0), \tag{1.52}$$

where $C(t,u_0,\theta_0)$ is a constant depending on t and the initial data.

The following fact will be used to prove (1.52), for any $\beta>\frac{1}{2}$,

$$\|\Lambda_{x_1}^{\frac{1}{2}}(u\cdot\nabla\theta)\|_{L^2}\leq C\|u\|_{H^1}(\|\nabla\theta\|_{L^2}+\|\Lambda_{x_1}^{\beta}\nabla\theta\|_{L^2}). \tag{1.53}$$

The Sketch Proof: $\alpha+\beta \geq 1$, $\beta > \frac{2+\sqrt{2}}{4}$

The key bound (1.52) along with $\beta > \frac{2+\sqrt{2}}{4}$ yields

Proposition 8

Assume that $(u_0, \theta_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ *with* $s > 2$ *and* $\nabla \cdot u_0 = 0$. *Assume* α *and* β *satisfy*

$$\alpha+\beta \geq 1, \qquad \beta > \frac{2+\sqrt{2}}{4}.$$

Then the corresponding solution of (1.29) obeys the following global bound

$$\|u(t)\|_{H^s}^2 + \|\theta(t)\|_{H^s}^2 + \int_0^t (\|\Lambda_{x_1}^{\alpha} u(\tau)\|_{H^s}^2 + \|\Lambda_{x_1}^{\beta} \theta(\tau)\|_{H^s}^2)\, d\tau < \infty.$$