

# Theory on Well-posedness of Boussinesq Equations with Fractional Laplacian

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# Sub-critical Dissipation with Lipschitz Data

## └ Fractional dissipation case

We consider the Cauchy problem of 2D fractional diffusion Boussinesq equations for an incompressible fluid flows in  $\mathbb{R}^2$

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nu(-\Delta)^\alpha u + \nabla P = \theta e_2, \\ \frac{\partial \theta}{\partial t} + (u \cdot \nabla)\theta + \kappa(-\Delta)^\beta \theta = 0, \\ \operatorname{div} u = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x). \end{cases} \quad (1)$$

where  $\alpha, \beta \in (0, 1)$ , and  $(-\Delta)^\alpha$  is the pseudodifferential operator defined via the Fourier transform

$$\widehat{(-\Delta)^\alpha v}(\xi) = |\xi|^{2\alpha} \widehat{v}(\xi).$$

In the following, for simplicity, we denote

$$\Lambda = (-\Delta)^{1/2}.$$

## Theorem 3.4 (Global well-posedness; Xiaojing Xu 2010)

Let  $\nu > 0$ ,  $\kappa > 0$  be fixed,  $\alpha \in [\frac{1}{2}, 1)$ ,  $\beta \in (0, \frac{1}{2}]$ ,  $\alpha + \beta = 1$ , and  $\operatorname{div} u_0 = 0$ . Let  $m > 2$  be an integer, and  $(u_0, \theta_0) \in H^m(\mathbb{R}^2)$ .

Then, there exists a unique solution  $(u, \theta)$  to the Cauchy problem (1) such that

$$\theta \in C([0, \infty); H^m(\mathbb{R}^2)) \cap L^2(0, \infty; H^{m+\beta}(\mathbb{R}^2)),$$

and

$$u \in C([0, \infty); H^m(\mathbb{R}^2)) \cap L^2(0, \infty; H^{m+\alpha}(\mathbb{R}^2)).$$

Remark 1.1. For simplicity of the exposition, we formulate and prove Theorem 3.4 in the subcritical case  $\alpha + \beta = 1$ , only. One can easily verify that, by arguments from this work, we can obtain an analogous result for  $1 \leq \alpha + \beta \leq 2$ .

### Lemma 3.1 (Positive Inequality)

Let  $0 \leq \alpha \leq 2$ . For every  $p > 1$ , we have

$$\int_{\mathbb{R}^n} (\Lambda^\alpha w) |w|^{p-2} w \, dx \geq C(p) \int_{\mathbb{R}^n} \left( \Lambda^{\frac{\alpha}{2}} |w|^{\frac{p}{2}} \right)^2 \, dx, \quad (2)$$

for all  $w \in L^p(\mathbb{R}^n)$  such that  $\Lambda^\alpha w \in L^p(\mathbb{R}^n)$ , where  $C(p) = \frac{4(p-1)}{p^2}$ .

**Remark:** This inequality is well-known in the theory of sub-Markovian operators e.g.

- V.A. Liskevich, Yu.A. Semenov, *Some problems on Markov semigroups, Schrödinger operators, Markov semigroups, wavelet analysis, operator algebras*, Akademie Verlag, Berlin, 1996.

Observe that if  $\alpha = 2$ , integrating by parts we obtain (2) with the equality.

## Theorem 3.5 (Blow-up Criterion; Xiaojing Xu 2010)

Let  $\alpha, \beta \in [0, 2]$ ,  $\nu \geq 0$ ,  $\kappa \geq 0$ . Suppose  $(u_0, \theta_0) \in H^m(\mathbb{R}^2)$  with  $m > 2$  being an integer. Then, there exists a unique local classical solution  $(u, \theta) \in C([0, T]; H^m(\mathbb{R}^2))$  of problem (1) for some  $T = T(\|u_0\|_{H^m(\mathbb{R}^2)}, \|\theta_0\|_{H^m(\mathbb{R}^2)})$ . Moreover, the solution remains in  $H^m(\mathbb{R}^2)$  up to a time  $T_1 > T$ , namely  $(u, \theta) \in C([0, T_1]; H^m(\mathbb{R}^2))$  if and only if

$$\int_0^T \|\nabla \theta(\tau)\|_{L^\infty} d\tau < \infty. \quad (3)$$

- In the inviscid case  $\nu = 0$  and  $\kappa = 0$ , Blow up Criterion was proved in [Chae1997](#). The arguments from that works with minor changes also for problem (1) with the fractional diffusion, due to inequality (2). By this reason, we skip details of the proof of Theorem 3.5.
- In order to prove Theorem 3.4, it suffices to show that (3) holds true for the smooth solutions  $(u, \theta)$  to the Cauchy problem (1).
- In the following section, we first show some a priori estimates for a smooth solution  $(u, \theta) \in C([0, T]; H^m(\mathbb{R}^2))$  with  $m > 2$  to (1), then prove that (3) is valid.

## └ Fractional dissipation case

For simplicity, let  $\nu = \kappa = 1$ .

- Estimate of  $\|\theta\|_{L^\infty(0, \infty; L^p(\mathbb{R}^2))}$ .

Let  $p \geq 2$ . Multiplying the second equation in (1) by  $|\theta|^{p-2}\theta$  and integrating over  $\mathbb{R}^2$ , we deduce that

$$\frac{1}{p} \frac{d}{dt} \|\theta(t)\|_{L^p}^p + \int_0^t (-\Delta)^\beta \theta |\theta|^{p-2} \theta dx = 0,$$

where we have used the divergence free condition. This identity together with Lemma 3.1, allows us to get

$$\|\theta(t)\|_{L^p}^p + C(p) \int_0^t \|\Lambda^\beta |\theta|^{\frac{p}{2}}(\tau)\|_{L^2}^2 d\tau \leq \|\theta_0\|_{L^p}^p.$$

In particular, when  $p = 2$ , we have

$$\|\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\beta \theta(\tau)\|_{L^2}^2 d\tau \leq \|\theta_0\|_{L^2}^2. \quad (4)$$



## └ Fractional dissipation case

- Estimate of  $\|u\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))}$ .

Multiplying the first equation of (1) by  $u$ , and integrating it over  $\mathbb{R}^2$ , we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + \int_{\mathbb{R}^2} u(-\Delta)^\alpha u \, dx &= \int_{\mathbb{R}^2} \theta e_2 |u|^2 \, dx - \int_{\mathbb{R}^2} (u \cdot \nabla) |u|^2 \, dx \\ &\quad - \int_{\mathbb{R}^2} \nabla P u \, dx. \end{aligned}$$

This identity together with inequality (2) and the divergence free condition, yield that

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + \int_{\mathbb{R}^2} |\Lambda^\alpha u|^2 \, dx = \int_{\mathbb{R}^2} \theta e_2 u \, dx \leq \|\theta(t)\|_{L^2} \|u(t)\|_{L^2}.$$

By (4) and the Hölder inequality, we deduce that

$$\|u(t)\|_{L^2}^2 + 4 \int_0^t \|\Lambda^\alpha u(\tau)\|_{L^2}^2 \, d\tau \leq 4 \|\theta_0\|_{L^2}^2 T^2 + 2 \|u_0\|_{L^2}^2.$$

## └ Fractional dissipation case

- Estimate of  $\|\omega(t)\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))}$ .

Taking the operation curl on both sides of the first equation in (1) and denoting  $\omega = \text{curl } u = \partial_{x_1} u_2 - \partial_{x_2} u_1$ , we get

$$\omega_t + (-\Delta)^\alpha \omega + (u \cdot \nabla) \omega = -\theta_{x_1}. \quad (5)$$

Multiplying the above equality by  $\omega$ , integrating over  $\mathbb{R}^2$ , we find

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega(t)\|_{L^2}^2 + \|\Lambda^\alpha \omega\|_{L^2}^2 &= \frac{1}{2} \int_{\mathbb{R}^2} (u \cdot \nabla) |\omega|^2 \, dx - \int_{\mathbb{R}^2} \theta_{x_1} \omega \, dx \\ &= - \int_{\mathbb{R}^2} \theta_{x_1} \omega \, dx \\ &\leq \frac{1}{2} \|\Lambda^\alpha \omega\|_{L^2}^2 + \frac{1}{2} \|\Lambda^\beta \theta\|_{L^2}^2 \end{aligned}$$

Here, in the last inequality, we have used the Parseval theorem and the relation  $\alpha + \beta = 1$ .

Thus, we have

$$\frac{d}{dt} \|\omega(t)\|_{L^2}^2 + \|\Lambda^\alpha \omega\|_{L^2}^2 \leq \|\Lambda^\beta \theta\|_{L^2}^2.$$

By virtue of estimate (4), we deduce that

$$\begin{aligned} \|\omega(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\alpha \omega(\tau)\|_{L^2}^2 d\tau &\leq \|\omega_0\|_{L^2}^2 + \int_0^t \|\Lambda^\beta \theta(\tau)\|_{L^2}^2 d\tau \\ &\leq C(\|\omega_0\|_{L^2}, \|\theta_0\|_{L^2}, T). \end{aligned} \tag{6}$$

## └ Fractional dissipation case

- Estimate of  $\|\Lambda^\alpha D\omega\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))}$

We first compute the derivative  $\nabla = (\partial_{x_1}, \partial_{x_2})$  of both sides of (5), and then take  $L^2$  inner product with  $\nabla\omega$ . After integration by parts, we obtain

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \|\nabla\omega(t)\|_{L^2}^2 + \|\Lambda^\alpha \nabla\omega\|_{L^2}^2 &= - \int_{\mathbb{R}^2} [\nabla(u \cdot \nabla)\omega] \nabla\omega dx - \int_{\mathbb{R}^2} \nabla\theta_{x_1} \nabla\omega dx \\
 &= - \int_{\mathbb{R}^2} [(\nabla u \cdot \nabla)\omega] \nabla\omega dx - \int_{\mathbb{R}^2} \nabla\theta_{x_1} \nabla\omega dx \\
 &\leq \|\nabla u\|_{L^{\frac{2}{1-\alpha}}} \|\nabla\omega\|_{L^2} \|\nabla\omega\|_{L^{\frac{2}{\alpha}}} + \frac{1}{2} \|\Lambda^\alpha \nabla\omega\|_{L^2}^2 + \frac{1}{2} \|\Lambda^\beta \nabla\theta\|_{L^2}^2 \\
 &\leq \|\nabla u\|_{L^{\frac{2}{1-\alpha}}}^{\frac{3\alpha-1}{2}} \|\nabla\omega\|_{L^2}^{\frac{3\alpha-1}{2}} \|\Lambda^\alpha \nabla\omega\|_{L^2}^{\frac{1-\alpha}{2}} + \frac{1}{2} \|\Lambda^\alpha \nabla\omega\|_{L^2}^2 + \frac{1}{2} \|\Lambda^\beta \nabla\theta\|_{L^2}^2 \\
 &\leq C \|\nabla u\|_{L^{\frac{2}{1-\alpha}}}^{\frac{2\alpha}{3\alpha-1}} \|\nabla\omega\|_{L^2}^2 + \frac{3}{4} \|\Lambda^\alpha \nabla\omega\|_{L^2}^2 + \frac{1}{2} \|\Lambda^\beta \nabla\theta\|_{L^2}^2
 \end{aligned} \tag{7}$$

where we have used the assumptions  $\alpha \geq \frac{1}{2}$  and  $\operatorname{div} u = 0$ .

## └ Fractional dissipation case

Next, computing the derivative  $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$  of the second equation from (1), we easily show that

$$\nabla^\perp \theta_t + \nabla^\perp [(u \cdot \nabla) \theta] + (-\Delta)^\beta \nabla^\perp \theta = 0.$$

We multiply the above equality by  $\nabla^\perp \theta$ , and integrate it over  $\mathbb{R}^2$ . Similar arguments as those in (7) lead to

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla^\perp \theta(t)\|_{L^2}^2 + \|\Lambda^\beta \nabla^\perp \theta\|_{L^2}^2 \\ & \leq - \int_{\mathbb{R}^2} (u \cdot \nabla) \nabla^\perp \theta \nabla^\perp \theta dx - \int_{\mathbb{R}^2} \nabla^\perp \theta \cdot \nabla u \nabla^\perp \theta dx \\ & = \|\nabla u\|_{L^{\frac{2}{1-\alpha}}} \|\nabla^\perp \theta\|_{L^2} \|\nabla^\perp \theta\|_{L^{\frac{2}{\alpha}}} \\ & \leq \|\nabla u\|_{L^{\frac{2}{1-\alpha}}} \|\nabla^\perp \theta\|_{L^2} \|\Lambda^\beta \nabla^\perp \theta\|_{L^2} \\ & \leq \frac{1}{2} \|\nabla u\|_{L^{\frac{2}{1-\alpha}}}^2 \|\nabla^\perp \theta\|_{L^2}^2 + \frac{1}{2} \|\Lambda^\beta \nabla^\perp \theta\|_{L^2}^2. \end{aligned} \tag{8}$$

└ Fractional dissipation case

Now, combining (7) with (8), one can show the function

$$X(t) = \|\nabla \omega(t)\|_{L^2} + \|\nabla^\perp \theta(t)\|_{L^2}.$$

satisfies the inequality

$$\frac{d}{dt} X(t) \leq C(\|\nabla u\|_{L^{\frac{2}{1-\alpha}}}^{\frac{2\alpha}{3\alpha-1}} + \|\nabla u\|_{L^{\frac{2}{1-\alpha}}}^2) X(t).$$

Therefore, Gronwall's inequality and the embedding inequality and estimate (6) yield that

$$\begin{aligned} X(t) &\leq CX(0) \exp\left\{\int_0^t (\|\nabla u(\tau)\|_{L^{\frac{2}{1-\alpha}}}^{\frac{2\alpha}{3\alpha-1}} + \|\nabla u(\tau)\|_{L^{\frac{2}{1-\alpha}}}^2) d\tau\right\} \\ &\leq CX(0) \exp\left\{\int_0^t (\|\Lambda^\alpha \omega(\tau)\|_{L^2}^{\frac{2\alpha}{3\alpha-1}} + \|\Lambda^\alpha \omega(\tau)\|_{L^2}^2) d\tau\right\} \\ &\leq CX(0) \exp\left\{T^{\frac{4\alpha-2}{3\alpha-1}} \left(\int_0^t \|\Lambda^\alpha \omega(\tau)\|_{L^2}^2 d\tau\right)^{\frac{2\alpha}{3\alpha-1}} + \int_0^t \|\Lambda^\alpha \omega(\tau)\|_{L^2}^2 d\tau\right\} \\ &\leq C(T, \|u_0\|_{H^2}, \|\theta_0\|_{H^1}). \end{aligned}$$

Finally, by virtue of estimate (7), we deduce

$$\|\nabla\omega(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\alpha \nabla\omega(\tau)\|_{L^2}^2 \, d\tau \leq C(T, \|u_0\|_{H^2}, \|\theta_0\|_{H^1}).$$

## └ Fractional dissipation case

- Estimate of  $\|\nabla\theta\|_{L^\infty(0,\infty;L^\infty(\mathbb{R}^2))}$

Multiplying (3.9) by  $|\nabla^\perp\theta|^{p-2}\nabla^\perp\theta$ , and integrating it over  $\mathbb{R}^2$ , we have

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|\nabla^\perp\theta(t)\|_{L^p}^p + C(p) \|\nabla^\perp\theta\|_{L^{\frac{2p}{1-\alpha}}}^p &\leq \int_{\mathbb{R}^2} \nabla^\perp\theta \cdot \nabla u |\nabla^\perp\theta|^{p-2} \nabla^\perp\theta dx \\ &\leq \|\nabla u\|_{L^\infty} \|\nabla^\perp\theta\|_{L^p}^p. \end{aligned}$$

Using Gronwall's inequality and the obvious identity  $\|\nabla Q\|_{L^p} = \|\nabla^\perp Q\|_{L^p}$ , we easily show that

$$\|\nabla\theta(t)\|_{L^p} \leq C \|\nabla\theta_0\|_{L^p} \exp \left\{ \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau \right\}.$$



# Fractional dissipation case

This inequality together with the Gagliardo-Nirenberg inequality, allows us to obtain

$$\begin{aligned}
 \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau &\leq C \int_0^t \|\omega(\tau)\|_{L^2}^{\frac{\alpha}{1+\alpha}} \left\| \Lambda^{1+\alpha} \omega(\tau) \right\|_{L^2}^{\frac{1}{1+\alpha}} d\tau \\
 &\leq CT^{\frac{2\alpha+1}{\alpha+1}} \left( \int_0^t \|\omega(\tau)\|_{L^2}^2 d\tau \right)^{\frac{\alpha}{1+2\alpha}} + C \int_0^t \left\| \Lambda^{1+\alpha} \omega(\tau) \right\|_{L^2}^2 d\tau \\
 &\leq C(T, \|u_0\|_{H^2}, \|\theta_0\|_{H^1}).
 \end{aligned}$$

Using Sobolev embedding

$$\|f(x)\|_{L^p(\mathbb{R}^2)} \leq C \|f(x)\|_{H^s(\mathbb{R}^2)}, \quad s > 1,$$

where  $C$  is independent of  $p \in [2, \infty]$ , then we have, for all  $t \in [0, T]$ ,

$$\|\nabla \theta(t)\|_{L^p} \leq C \|\theta_0\|_{H^m} \exp \left\{ \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau \right\} \leq C,$$

where  $C$  is independent of  $p$ .

Passing to the limit  $p \rightarrow \infty$  in above inequality, we obtain

$$\|\nabla\theta(t)\|_{L^\infty} \leq C(T, \|u_0\|_{H^2}, \|\theta_0\|_{H^m}), \quad \forall t \in [0, T].$$

This implies that condition (3) holds true, and according to Theorem 3.5, we obtain a unique solution of (1) such that  $(u, \theta) \in C([0, \infty); H^m(\mathbb{R}^2))$ . By (7), (8) and the iteration process, we construct  $u \in L^2(0, \infty; H^{m+\alpha}(\mathbb{R}^2))$  and  $\theta \in L^2(0, \infty; H^{m+\beta}(\mathbb{R}^2))$ , and we complete the proof of Theorem 3.4.  $\square$

# The Sub-critical Dissipation with Yudovich Type Data

## Global well-posedness

## Sub-critical Boussinesq Equations with Yudovich Initial Data

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu \Lambda^\alpha u = -\nabla p + \theta e_2, \\ \partial_t \theta + u \cdot \nabla \theta + \Lambda^\beta \theta = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x), \end{cases} \quad (9)$$

where  $\nu \geq 0$ ,  $\alpha \in (0, 1]$  and  $\beta \in (1, 2]$ .

## Theorem 1 (Wu-X., 2014)

Consider (9) with either  $\nu > 0$  or  $\nu = 0$ ,  $\alpha \in (0, 1]$  and  $\beta \in (1, 2]$ . Let  $q > \frac{2}{\beta-1}$ . Assume  $u_0$  satisfying  $\nabla \cdot u_0 = 0$  and  $u_0 \in L^2(\mathbb{R}^2)$ ,  $\omega_0 \equiv \nabla \times u_0 \in L^q(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ . Assume  $\theta_0 \in L^2(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ . Then, there exists a unique solution  $(u, \theta)$  to (9) such that, for some  $r \in (2, q)$ ,

$$\begin{aligned} u &\in C_{loc}([0, \infty); L^2 \cap L^\infty), \quad \omega \in L_{loc}^\infty(0, \infty; L^q \cap L^\infty), \\ \theta &\in C([0, \infty); L^2 \cap L^q) \cap L^2(0, \infty; H^{\frac{\beta}{2}}) \cap L_{loc}^1(0, \infty; B_{r,1}^{1+\frac{2}{r}}(\mathbb{R}^2)) \end{aligned} \quad (10)$$

Furthermore, the bounds for  $(u, \theta, \omega)$  in the class (10) are independent of  $\nu$  even in the case when  $\nu > 0$ .

## Lemma 1

Assume that  $\nabla \cdot u = 0$  and  $\theta$  solves

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \Lambda^\beta \theta = 0, \\ \theta(x, 0) = \theta_0(x). \end{cases} \quad (11)$$

Let  $r \in [2, \infty)$ . Then  $\theta$  obeys the estimate: for any integer  $j \geq 0$ ,

$$2^{\beta j} \|\Delta_j \theta\|_{L_t^1 L^r} \leq C \|\theta_0\|_{L^r} \left( 1 + \|\omega\|_{L_t^1 L^q} + (j+2) \|\omega\|_{L_t^1 L^\infty} \right). \quad (12)$$

*Proof of Lemma.* Applying  $\Delta_j$  to (11) and taking inner product with  $\Delta_j \theta |\Delta_j \theta|^{r-2}$ , we obtain

$$\frac{1}{r} \frac{d}{dt} \|\Delta_j \theta\|_{L^r}^r + \int \Delta_j \theta |\Delta_j \theta|^{r-2} \Lambda^\beta \theta \, dx = - \int \Delta_j \theta |\Delta_j \theta|^{r-2} \Delta_j (u \cdot \nabla \theta) \, dx. \quad (13)$$

The dissipative term obeys the following lower bound

$$\int \Delta_j \theta |\Delta_j \theta|^{r-2} \Lambda^\beta \theta \, dx \geq C 2^{\beta j} \|\Delta_j \theta\|_{L^r}^r.$$

## └ Fractional dissipation case

Using the notion of para-products, we write

$$\Delta_j(u \cdot \nabla \theta) = J_{11} + J_{12} + J_{13} + J_{14} + J_{15},$$

where

$$J_{11} = \sum_{|j-k| \leq 2} [\Delta_j, S_{k-1} u \cdot \nabla] \Delta_k \theta,$$

$$J_{12} = \sum_{|j-k| \leq 2} (S_{k-1} u - S_j u) \cdot \nabla \Delta_j \Delta_k \theta,$$

$$J_{13} = S_j u \cdot \nabla \Delta_j \theta,$$

$$J_{14} = \sum_{|j-k| \leq 2} \Delta_j (\Delta_k u \cdot \nabla S_{k-1} \theta),$$

$$J_{15} = \sum_{k \geq j-1} \Delta_j (\Delta_k u \cdot \nabla \tilde{\Delta}_k \theta).$$



## └ Fractional dissipation case

Since  $\nabla \cdot u = 0$ , we have

$$\int J_{13} |\Delta_j \theta|^{r-2} \Delta_j \theta \, dx = 0.$$

By Hölder's inequality,

$$\left| \int J_{11} |\Delta_j \theta|^{r-2} \Delta_j \theta \, dx \right| \leq \|J_{11}\|_{L^r} \|\Delta_j \theta\|_{L^r}^{r-1}.$$

**Remark:** Introduce the Littlewood-Paley decomposition, we write for each  $j \in \mathbb{Z}$

$$A_j = \left\{ \xi \in \mathbb{R}^d : 2^{j-1} \leq |\xi| < 2^{j+1} \right\}.$$

The Littlewood-Paley decomposition asserts the existence of a sequence of functions  $\{\Phi_j\}_{j \in \mathbb{Z}} \in \mathcal{S}$  such that

$$\text{supp } \widehat{\Phi}_j \subset A_j, \quad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd} \Phi_0(2^j x),$$

## └ Fractional dissipation case

We write the commutator in terms of the integral,

$$J_{11} = \int \Phi_j(x-y) (S_{k-1}u(y) - S_{k-1}u(x)) \cdot \nabla \Delta_k \theta(y) dy,$$

where  $\Phi_j$  is the kernel of the operator  $\Delta_j$ . By Young's inequality for convolution,

$$\begin{aligned} \|J_{11}\|_{L^r} &\leq \| |x| \Phi_j(x) \|_{L^1} \|\nabla S_{j-1}u\|_{L^\infty} \|\nabla \Delta_j \theta\|_{L^r} \\ &\leq \| |x| \Phi_0(x) \|_{L^1} \|\nabla S_{j-1}u\|_{L^\infty} \|\Delta_j \theta\|_{L^r} \\ &= C \|\nabla S_{j-1}u\|_{L^\infty} \|\Delta_j \theta\|_{L^r}, \end{aligned}$$

where we have used the definition of  $\Phi_j$  and Bernstein's inequality in the second inequality above.

## └ Fractional dissipation case

By Bernstein's inequality,

$$\begin{aligned}\|J_{12}\|_{L^r} &\leq C \|\Delta_j u\|_{L^\infty} \|\nabla \Delta_j \theta\|_{L^r} \\ &\leq C \|\nabla \Delta_j u\|_{L^\infty} \|\Delta_j \theta\|_{L^r} \\ &\leq C \|\omega\|_{L^\infty} \|\Delta_j \theta\|_{L^r};\end{aligned}$$

$$\begin{aligned}\|J_{14}\|_{L^r} &\leq C \|\Delta_j u\|_{L^\infty} \|\nabla S_{j-1} \theta\|_{L^r} \\ &\leq C \|\nabla \Delta_j u\|_{L^\infty} \sum_{m \leq j-2} 2^{(m-j)} \|\Delta_m \theta\|_{L^r} \\ &\leq C \|\omega\|_{L^\infty} \sum_{m \leq j-2} 2^{(m-j)} \|\Delta_m \theta\|_{L^r};\end{aligned}$$

$$\begin{aligned}\|J_{15}\|_{L^r} &\leq C \sum_{k \geq j-1} 2^{(j-k)} \|\nabla \Delta_k u\|_{L^\infty} \|\Delta_k \theta\|_{L^r} \\ &\leq C \|\omega\|_{L^\infty} \sum_{k \geq j-1} 2^{(j-k)} \|\Delta_k \theta\|_{L^r}.\end{aligned}$$

Inserting the estimates above in (13), we have

$$\begin{aligned} \frac{d}{dt} \|\Delta_j \theta\|_{L^r} + C 2^{\beta j} \|\Delta_j \theta\|_{L^r} &\leq C \|\nabla S_{j-1} u\|_{L^\infty} \|\Delta_j \theta\|_{L^r} \\ &\quad + C \|\omega\|_{L^\infty} [\|\Delta_j \theta\|_{L^r} + \sum_{m \leq j-2} 2^{(m-j)} \|\Delta_m \theta\|_{L^r} \\ &\quad + \sum_{k \geq j-1} 2^{(j-k)} \|\Delta_k \theta\|_{L^r}]. \end{aligned}$$

Integrating in time, taking  $L_t^1$  and multiplying by  $2^{\beta j}$ , we obtain (12). This completes the proof of Lemma 1.

# Proof of Theorem 1

The local existence can be established through a standard procedure. We provide the global *a priori* bounds needed for the global existence. Obviously,

$$\begin{aligned}\|\theta(t)\|_{L^2} &\leq \|\theta_0\|_{L^2}, \quad \|\theta(t)\|_{L^q} \leq \|\theta_0\|_{L^q}, \\ \|u(t)\|_{L^2} &\leq \|u_0\|_{L^2} + t \|\theta_0\|_{L^2}.\end{aligned}$$

Since  $\omega$  satisfies

$$\partial_t \omega + u \cdot \nabla \omega + \nu \Lambda^\alpha \omega = \partial_{x_1} \theta,$$

it is clear that, for any  $\nu \geq 0$ ,

$$\|\omega(t)\|_{L^q \cap L^\infty} \leq \|\omega_0\|_{L^q \cap L^\infty} + \int_0^t \|\partial_{x_1} \theta(\tau)\|_{L^q \cap L^\infty} d\tau. \quad (14)$$

└ Fractional dissipation case

To obtain a global bound for  $\|\omega(t)\|_{L^q \cap L^\infty}$ , we make use of the smoothing effect in the  $\theta$ -equation. By Lemma 1, for any  $2 < r < q$  and  $0 < \epsilon < \beta$ ,

$$\sup_{j \geq -1} 2^{(\beta-\epsilon)j} \|\Delta_j \theta\|_{L_t^1 L^r} \leq C \|\theta_0\|_{L^r} (1 + \|\omega\|_{L_t^1(L^\infty \cap L^q)}).$$

Choosing  $2 < r < q$  and  $0 < \epsilon < \beta$  such that  $1 + \frac{2}{r} < \beta - \epsilon$ , we have, by Bernstein's inequality,

$$\begin{aligned} \|\theta\|_{L_t^1 B_{r,1}^{1+\frac{2}{r}}} &\leq \sup_{j \geq -1} 2^{(\beta-\epsilon)j} \|\Delta_j \theta\|_{L_t^1 L^r}, \\ \|\nabla \theta(\tau)\|_{L^q \cap L^\infty} &\leq \sum_{j \geq -1} 2^j \|\Delta_j \theta\|_{L^q \cap L^\infty} \\ &\leq C \sum_{j \geq -1} 2^{(1+\frac{2}{r})j} \|\Delta_j \theta\|_{L^r} = C \|\theta\|_{B_{r,1}^{1+\frac{2}{r}}}. \end{aligned} \tag{15}$$

└ Fractional dissipation case

Combining these estimates, we obtain

$$\|\theta\|_{L_t^1 B_{r,1}^{1+\frac{2}{r}}} \leq C \|\theta_0\|_{L^r} \left( 1 + t \|\omega_0\|_{L^\infty \cap L^q} + \int_0^t \|\theta\|_{L_\tau^1 B_{r,1}^{1+\frac{2}{r}}} d\tau \right).$$

By Gronwall's inequality, for  $C$  depending on  $\|\theta_0\|_{L^2 \cap L^q}$  and  $\|\omega_0\|_{L^\infty \cap L^q}$  only,

$$\|\theta\|_{L_t^1 B_{r,1}^{1+\frac{2}{r}}} \leq C e^{Ct}.$$

Consequently, by (14),  $\|\omega\|_{L_t^\infty(L^\infty \cap L^q)} \leq C e^{Ct}$ .

In addition, by Gagliardo-Nirenberg inequality,

$$\|u\|_{L^\infty} \leq C \|u\|_{L^2}^{\frac{1}{2}} \|\omega\|_{L^\infty}^{\frac{1}{2}},$$

which yields the global bound for  $\|u\|_{L^\infty}$ . This completes the proof for the global bounds, which are independent of  $\nu$  even in the case when  $\nu > 0$ .

Next we show that any two solutions satisfying (10) must coincide. Assume that  $(u_1, \theta_1)$  and  $(u_2, \theta_2)$  are two solutions of (9) and let  $p_1$  and  $p_2$  be the associated pressures, respectively. Consider the differences

$$u = u_2 - u_1, \quad \theta = \theta_2 - \theta_1, \quad p = p_2 - p_1,$$

which satisfy

$$\begin{aligned} \partial_t u + u \cdot \nabla u_2 + u_1 \cdot \nabla u + \nu \Lambda^\alpha u &= -\nabla p + \theta e_2, \\ \partial_t \theta + u \cdot \nabla \theta_2 + u_1 \cdot \nabla \theta + \Lambda^\beta \theta &= 0. \end{aligned}$$



Taking the inner product with  $(u, \theta)$  and integrating by parts lead to

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2 + \|\theta\|_{L^2}^2) + \nu \|\Lambda^{\frac{\alpha}{2}} u\|_{L^2}^2 + \|\Lambda^{\frac{\beta}{2}} \theta\|_{L^2}^2 \\ & \leq \|u\|_{L^2} \|\theta\|_{L^2} + J_1 + J_2, \end{aligned} \quad (16)$$

where

$$J_1 = - \int u \cdot \nabla u_2 \cdot u \, dx, \quad J_2 = - \int u \cdot \nabla \theta_2 \cdot \theta \, dx.$$

For notational convenience, we let  $\delta > 0$  and write

$$Y_\delta^2(t) = \delta^2 + \|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2.$$

## └ Fractional dissipation case

For any  $\rho \in [2, \infty)$ , we have

$$|J_1| \leq \|u\|_{L^\infty}^{\frac{2}{\rho}} \|\nabla u_2\|_{L^\rho} \|u\|_{L^2}^{2-\frac{2}{\rho}}.$$

Furthermore,

$$\begin{aligned} \|u\|_{L^\infty} &\leq \|u_1\|_{L^\infty} + \|u_2\|_{L^\infty} = L(t), \\ \|\nabla u_2\|_{L^\rho} &\leq \rho \sup_{\rho \geq 2} \frac{\|\nabla u_2\|_{L^\rho}}{\rho} \leq \rho M(t) \end{aligned}$$

where we have used the fact that  $\sup_{\rho \geq 2} \frac{\|\nabla u_2\|_{L^\rho}}{\rho}$  is bounded due to  $\omega_2 \in L^q \cap L^\infty$ .

## └ Fractional dissipation case

Therefore, by optimizing the bound over  $\rho$ , we have

$$|J_1| \leq M(t) \rho \left( \frac{L(t)}{Y_\delta} \right)^{\frac{2}{\rho}} Y_\delta^2 \leq 2e M(t) [\log L(t) - \log Y_\delta(t)] Y_\delta^2(t).$$

To bound  $J_2$ , we recall (15) to get

$$|J_2| \leq \|\nabla \theta_2\|_{L^\infty} \|u\|_{L^2} \|\theta\|_{L^2} \leq \|\theta_2\|_{B_{r,1}^{1+\frac{1}{r}}} Y_\delta^2(t).$$

Inserting the bounds above in (16), we obtain

$$\frac{d}{dt} Y_\delta(t) \leq 2e M(t) [\log L(t) - \log Y_\delta(t)] Y_\delta(t) + (1 + \|\theta_2\|_{B_{r,1}^{1+\frac{1}{r}}}) Y_\delta(t).$$

It then follows from applying the Osgood inequality that

$$Y_\delta(t) \leq L(t)^{1-B(t)} Y_\delta(0)^{B(t)},$$

where

$$B(t) \equiv \exp \left( - \int_0^t (2e M(\tau) + \|\theta_2(\tau)\|_{B_{r,1}^{1+\frac{1}{r}}}) d\tau \right).$$

Since  $Y_\delta(0) = \delta$ , we obtain by letting  $\delta \rightarrow 0$  that

$$\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 \equiv 0$$

for any  $t > 0$ . This proves the uniqueness and thus Theorem 1.

# The Critical Dissipation

The Cauchy problem of 2D fractional Boussinesq equations

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nu \Lambda^\alpha u + \nabla P = \theta e_2, \\ \frac{\partial \theta}{\partial t} + (u \cdot \nabla)\theta + \kappa \Lambda^\beta \theta = 0, \\ \operatorname{div} u = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x). \end{cases} \quad (17)$$

where  $u(t, x) = (u_1, u_2)$  is the velocity vector field,  $P = P(t, x)$  is the scalar pressure,  $\theta(t, x)$  is the scalar temperature,  $\nu \geq 0$  is the viscosity,  $\kappa \geq 0$  is the thermal diffusivity,  $e_2 = (0, 1)$ ,  $\alpha, \beta \in (0, 2)$ .

$$\widehat{(-\Delta)^\alpha v}(\xi) = |\xi|^{2\alpha} \widehat{v}(\xi).$$

In the following, for simplicity, we denote

$$\Lambda = (-\Delta)^{1/2}.$$

## Two special critical cases

Hmidi, Keraani and Rousset were able to establish the global regularity of the critical Navier-Stokes-Boussinesq equations, namely (1) with  $\nu > 0$ ,  $\kappa = 0$  and  $\alpha = 1$ .

- [T. Hmidi](#), [S. Keraani](#) and [F. Rousset](#), Global well-posedness for a Boussinesq Navier-Stokes system with critical dissipation, J. Differential Equations 249 (2010).

## Theorem 2 (Hmidi, Keraani and Rousset 2010)

*Let  $\theta^0 \in L^2 \cap B_{\infty,1}^0$  and  $v^0$  be a divergence-free vector field belonging to  $H^1 \cap \dot{W}^{1,p}$  with  $p \in (2, +\infty)$ . Then the system has a unique global solution  $(v, \theta)$  such that*

$$v \in L_{\text{loc}}^\infty(\mathbb{R}_+; H^1 \cap \dot{W}^{1,p}) \cap L_{\text{loc}}^1(\mathbb{R}_+; B_{\infty,1}^1)$$

*and*

$$\theta \in L_{\text{loc}}^\infty(\mathbb{R}_+; L^2 \cap B_{\infty,1}^0).$$



**Motivation:** Consider the new function

$$\omega - \mathcal{R}\theta.$$

The vorticity equation is given by

$$\omega_t + u \cdot \nabla \omega + \Lambda \omega = \theta_{x_1}, \quad \Lambda = (-\Delta)^{\frac{1}{2}}.$$

The idea is to write  $\Lambda \omega - \theta_{x_1} = \Lambda(\omega - \mathcal{R}\theta)$ ,  $\mathcal{R} = \Lambda^{-1} \partial_{x_1}$  and consider the difference with  $(\mathcal{R}\theta)_t + \vec{u} \cdot \nabla(\mathcal{R}\theta) = -[\mathcal{R}, u \cdot \nabla]\theta$ ,

$$(\omega - \mathcal{R}\theta)_t + u \cdot \nabla(\omega - \mathcal{R}\theta) + \nu \Lambda(\omega - \mathcal{R}\theta) = [\mathcal{R}, u \cdot \nabla]\theta.$$

The advantage of this new equation is that the commutator is much more regular and thus can be controlled. In fact, the  $L^q$ -norm of this commutator is more or less bound by  $\|\nabla u\|_{L^q} \|\theta\|_{B_{\infty,1}^0}$  and thus this new formation makes the global  $L^q$  bound for  $\omega$  possible.

## └ Fractional dissipation case

In addition, Hmidi, Keraani and Rousset were also able to establish the global regularity of the critical Euler-Boussinesq equations, namely (1) with  $\nu = 0, \kappa > 0$  and  $\beta = 1$ .

### Theorem 3 (Hmidi, Keraani and Rousset 2011)

*Let  $p \in (2, \infty)$ ,  $v^0 \in B_{\infty,1}^1 \cap \dot{W}^{1,p}$  be a divergence-free vector field of  $\mathbb{R}^2$  and  $\theta^0 \in B_{\infty,1}^0 \cap L^p$ . Then there exists a unique global solution  $(v, \theta)$  to the system with*

$$v \in L_{\text{loc}}^\infty(\mathbb{R}_+; B_{\infty,1}^1 \cap \dot{W}^{1,p}), \quad \theta \in L_{\text{loc}}^\infty(\mathbb{R}_+; B_{\infty,1}^0 \cap L^p) \cap \tilde{L}_{\text{loc}}^1(\mathbb{R}_+; B_{p,\infty}^1).$$

T. Hmidi, S. Keraani and F. Rousset, Global well-posedness for Euler-Boussinesq system with critical dissipation, Comm. Partial Differential Equations 36 (2011).

# Proof of Theorem 3

We remark that direct energy estimates will not allow one to obtain even global  $L^2$  bound on the vorticity. Their idea is to consider the evolution of the new function  $\omega + \mathcal{R}\theta$ . *a priori* estimates.

One key ingredient is to combine the vorticity equation

$$\omega_t + u \cdot \nabla \omega = \theta_{x_1}$$

with the equation for  $\theta$ ,

$$(\mathcal{R}\theta)_t + u \cdot \nabla(\mathcal{R}\theta) = -\mathcal{R}\Lambda\theta - [\mathcal{R}, u \cdot \nabla]\theta$$

into the equation

$$(\omega + \mathcal{R}\theta)_t + u \cdot \nabla(\omega + \mathcal{R}\theta) = -[\mathcal{R}, u \cdot \nabla]\theta.$$

## └ Fractional dissipation case

The main technical difficulty in this program when one takes the nonlinear terms into account is to evaluate the commutator  $[\mathcal{R}, v \cdot \nabla]$ . First we give some properties of the Riez operator  $\mathcal{R} = \partial_1/|D|$ , where the operator  $|D|^\alpha$  is defined by

$$\mathcal{F}(|D|^\alpha u)(\xi) = |\xi|^\alpha (\mathcal{F}u)(\xi).$$

### Proposition 4

*Let  $\mathcal{R}$  be the Riez operator  $\mathcal{R} = \partial_1/|D|$ . Then the following hold true.*

*i) For every  $p \in (1, +\infty)$ ,*

$$\|\mathcal{R}\|_{\mathcal{L}(L^p)} \lesssim 1.$$

## Proposition 4

ii) Let  $\chi \in \mathcal{D}(\mathbb{R}^d)$ . Then, there exists  $C > 0$  such that

$$\| |D|^s \chi(2^{-q}|D|) \mathcal{R} \|_{\mathcal{L}(L^p)} \leq C 2^{qs},$$

for every  $(p, s, q) \in [1, \infty] \times (0, +\infty) \times \mathbb{N}$ .

iii) Let  $\mathcal{C}$  be a fixed ring. Then, there exists  $\psi \in \mathcal{S}$  whose spectrum does not meet the origin such that

$$\mathcal{R}f = 2^{qd} \psi(2^q \cdot) * f$$

for every  $f$  with Fourier transform supported in  $2^q \mathcal{C}$ .

└ Fractional dissipation case

Controlling the commutator between  $\mathcal{R}$  and the convection operator  $v \cdot \nabla$  is a crucial ingredient in the proof of Theorem 3.

### Proposition 5 (Commutator estimate)

*Let  $v$  be is a smooth divergence-free vector field.*

*i) For every  $(p, r) \in [2, \infty) \times [1, \infty]$  there exists a constant  $C = C(p, r)$  such that*

$$\|[\mathcal{R}, v \cdot \nabla]\theta\|_{B_{p,r}^0} \leq C \|\nabla v\|_{L^p} (\|\theta\|_{B_{\infty,r}^0} + \|\theta\|_{L^p}),$$

*for every smooth scalar function  $\theta$ .*

*ii) For every  $(r, \rho) \in [1, \infty] \times (1, \infty)$  and  $\epsilon > 0$  there exists a constant  $C = C(r, \rho, \epsilon)$  such that*

$$\|[\mathcal{R}, v \cdot \nabla]\theta\|_{B_{\infty,r}^0} \leq C (\|\omega\|_{L^\infty} + \|\omega\|_{L^\rho}) (\|\theta\|_{B_{\infty,r}^\epsilon} + \|\theta\|_{L^\rho}),$$

*for every smooth scalar function  $\theta$ .*

The following inequality will be useful to give the proof of Proposition 5.

### Lemma 6

*Given  $(p, m) \in [1, \infty]^2$  such that  $p \geq m'$  with  $m'$  the conjugate exponent of  $m$ . Let  $f, g$  and  $h$  be three functions such that  $\nabla f \in L^p$ ,  $g \in L^m$  and  $xh \in L^{m'}$ . Then,*

$$\|h * (fg) - f(h * g)\|_{L^p} \leq \|xh\|_{L^{m'}} \|\nabla f\|_{L^p} \|g\|_{L^m}.$$

## Commutator estimate

We split the commutator into three parts,

$$\begin{aligned}
 [\mathcal{R}, v \cdot \nabla] \theta &= \sum_{q \in \mathbb{N}} [\mathcal{R}, S_{q-1} v \cdot \nabla] \Delta_q \theta + \sum_{q \in \mathbb{N}} [\mathcal{R}, \Delta_q v \cdot \nabla] S_{q-1} \theta \\
 &\quad + \sum_{q \geq -1} [\mathcal{R}, \Delta_q v \cdot \nabla] \tilde{\Delta}_q \theta \\
 &= \sum_{q \in \mathbb{N}} \text{I}_q + \sum_{q \in \mathbb{N}} \text{II}_q + \sum_{q \geq -1} \text{III}_q \\
 &= \text{I} + \text{II} + \text{III}.
 \end{aligned}$$



## └ Fractional dissipation case

We start with the estimate of the first term I. According to the *iii)* of Proposition 4 there exists  $h \in \mathcal{S}$  whose spectrum does not meet the origin such that

$$I_q(x) = h_q * (S_{q-1}v \cdot \nabla \Delta_q \theta) - S_{q-1}v \cdot (h_q * \nabla \Delta_q \theta),$$

where  $h_q(x) = 2^{dq} h(2^q x)$ .

Applying Lemma 6 ( $\|h * (fg) - f(h * g)\|_{L^p} \leq \|xh\|_{L^{m'}} \|\nabla f\|_{L^p} \|g\|_{L^m}$ ). with  $m = \infty$  we get

$$\begin{aligned} \|I_q\|_{L^p} &\lesssim \|xh_q\|_{L^1} \|\nabla S_{q-1}v\|_{L^p} \|\Delta_q \nabla \theta\|_{L^\infty} \\ &\lesssim \|\nabla v\|_{L^p} \|\Delta_q \theta\|_{L^\infty}. \end{aligned} \tag{18}$$

In the last line we've used Bernstein inequality and

$$\|xh_q\|_{L^1} = 2^{-q} \|xh\|_{L^1}.$$

Combined with the trivial fact

$$\Delta_j \sum_q \mathbf{I}_q = \sum_{|j-q| \leq 4} \mathbf{I}_q$$

this yields

$$\begin{aligned} \|\mathbf{I}\|_{B_{p,r}^0} &\lesssim \left( \sum_{q \geq -1} \|\mathbf{I}_q\|_{L^p}^r \right)^{\frac{1}{r}} \\ &\lesssim \|\nabla v\|_{L^p} \|\theta\|_{B_{\infty,r}^0}. \end{aligned}$$

Let us move to the second term II. As before one writes

$$\text{II}_q(x) = h_q * (\Delta_q v \cdot \nabla S_{q-1} \theta) - \Delta_q v \cdot (h_q * \nabla S_{q-1} \theta),$$

and then we obtain the estimate

$$\begin{aligned} \|\text{II}_q\|_{L^p} &\lesssim 2^{-q} \|\Delta_q \nabla v\|_{L^p} \|S_{q-1} \nabla \theta\|_{L^\infty} \\ &\lesssim \|\nabla v\|_{L^p} \sum_{j \leq q-2} 2^{j-q} \|\Delta_j \theta\|_{L^\infty}. \end{aligned}$$

Combined with convolution inequalities this yields

$$\|\text{II}\|_{B_{p,r}^0} \lesssim \|\nabla v\|_{L^p} \|\theta\|_{B_{\infty,r}^0}.$$

Let us now deal with the third term III. Using that the divergence of  $\Delta_q v$  vanishes, we rewrite III as

$$\begin{aligned}
 \text{III} &= \sum_{q \geq 2} \mathcal{R} \operatorname{div}(\Delta_q v \tilde{\Delta}_q \theta) - \sum_{q \geq 2} \operatorname{div}(\Delta_q v \mathcal{R} \tilde{\Delta}_q \theta) \\
 &\quad + \sum_{q \leq 1} [\mathcal{R}, \Delta_q v \cdot \nabla] \tilde{\Delta}_q \theta \\
 &= J_1 + J_2 + J_3.
 \end{aligned}$$

Using Proposition 4 *ii*), ( $\| |\mathbf{D}|^s \chi(2^{-q} |\mathbf{D}|) \mathcal{R} \|_{\mathcal{L}(L^p)} \leq C 2^{qs}$ ) we get

$$\left\| \Delta_j \mathcal{R} \operatorname{div}(\Delta_q v \tilde{\Delta}_q \theta) \right\|_{L^p} \lesssim 2^j \|\Delta_q v\|_{L^p} \|\tilde{\Delta}_q \theta\|_{L^\infty}.$$

└ Fractional dissipation case

Also, since  $\tilde{\Delta}_q \theta$  is supported away from zero for  $q \geq 2$  then Proposition 4 iii)  $(\mathcal{R}f = 2^{qd}\psi(2^q \cdot) * f)$  yields

$$\begin{aligned} \|\Delta_j \operatorname{div}(\Delta_q v \mathcal{R} \tilde{\Delta}_q \theta)\|_{L^p} &\lesssim 2^j \|\Delta_q v\|_{L^p} \|\mathcal{R} \tilde{\Delta}_q \theta\|_{L^\infty} \\ &\lesssim 2^j \|\Delta_q v\|_{L^p} \|\tilde{\Delta}_q \theta\|_{L^\infty}. \end{aligned}$$

Therefore we get

$$\begin{aligned} \|\Delta_j(J_1 + J_2)\|_{L^p} &\lesssim \sum_{\substack{q \in \mathbb{N} \\ q \geq j-4}} 2^j \|\Delta_q v\|_{L^p} \|\tilde{\Delta}_q \theta\|_{L^\infty} \\ &\lesssim \|\nabla v\|_{L^p} \sum_{\substack{q \in \mathbb{N} \\ q \geq j-4}} 2^{j-q} \|\Delta_q \theta\|_{L^\infty}, \end{aligned}$$

where we have again used Bernstein inequality to get the last line. It suffices now to use convolution inequalities to get

$$\|J_1 + J_2\|_{B_{p,r}^0} \lesssim \|\nabla v\|_{L^p} \|\theta\|_{B_{\infty,r}^0}.$$

For the last term  $J_3$  we can write

$$\sum_{-1 \leq q \leq 1} [\mathcal{R}, \Delta_q v \cdot \nabla] \tilde{\Delta}_q \theta(x) = \sum_{q \leq 1} [\operatorname{div} \tilde{\chi}(D) \mathcal{R}, \Delta_q v] \tilde{\Delta}_q \theta(x),$$

where  $\tilde{\chi}$  belongs to  $\mathcal{D}(\mathbb{R}^d)$ . Proposition 4 ensures that  $\operatorname{div} \tilde{\chi}(D) \mathcal{R}$  is a convolution operator with a kernel  $\tilde{h}$  satisfying

$$|\tilde{h}(x)| \lesssim (1 + |x|)^{-d-1}.$$

Details refer to see the proof of Proposition 3.1. [Hmidi, Keraani and Rousset 2011](#). Thus

$$J_3 = \sum_{q \leq 1} \tilde{h} * (\Delta_q v \cdot \tilde{\Delta}_q \theta) - \Delta_q v \cdot (\tilde{h} * \tilde{\Delta}_q \theta).$$

## └ Fractional dissipation case

First of all we point out that  $\Delta_j J_3 = 0$  for  $j \geq 6$ , thus we just need to estimate the low frequencies of  $J_3$ . Noticing that  $x\tilde{h}$  belongs to  $L^{p'}$  for  $p' > 1$  then using Lemma 6 ( $\|h * (fg) - f(h * g)\|_{L^p} \leq \|xh\|_{L^{m'}} \|\nabla f\|_{L^p} \|g\|_{L^m}$ ) with  $m = p \geq 2$  we obtain

$$\begin{aligned} \|\Delta_j J_3\|_{L^p} &\lesssim \sum_{q \leq 1} \|x\tilde{h}\|_{L^{p'}} \|\Delta_q \nabla v\|_{L^p} \|\tilde{\Delta}_q \theta\|_{L^p} \\ &\lesssim \|\nabla v\|_{L^p} \sum_{-1 \leq q \leq 1} \|\Delta_q \theta\|_{L^p}. \end{aligned}$$

This yields finally

$$\|J_3\|_{B_{p,r}^0} \lesssim \|\nabla v\|_{L^p} \|\theta\|_{L^p}.$$

This completes the proof of the first part of Proposition 5.

└ Fractional dissipation case

To estimate the terms I and II we use two facts: the first one is  $\|\Delta_q \nabla u\|_{L^\infty} \approx \|\Delta_q \omega\|_{L^\infty}$  for all  $q \in \mathbb{N}$ . The second one is

$$\begin{aligned} \|\nabla S_{q-1} v\|_{L^\infty} &\lesssim \|\nabla \Delta_{-1} v\|_{L^\infty} + \sum_{j=0}^{q-2} \|\Delta_j \nabla v\|_{L^\infty} \\ &\lesssim \|\omega\|_{L^\rho} + q \|\omega\|_{L^\infty}. \end{aligned}$$

For the remainder term we do the same analysis as before except for  $J_3$ : we apply Lemma 6 with  $p = \infty$  and  $m = \rho$  leading to

$$\begin{aligned} \|\Delta_j J_3\|_{L^p} &\lesssim \sum_{q \leq 1} \|x \tilde{h}\|_{L^{\rho'}} \|\Delta_q \nabla v\|_{L^\infty} \|\tilde{\Delta}_q \theta\|_{L^p} \\ &\lesssim \|\nabla v\|_{L^\rho} \sum_{-1 \leq q \leq 1} \|\Delta_q \theta\|_{L^p} \\ &\lesssim \|\omega\|_{L^\rho} \|\theta\|_{L^\rho}. \end{aligned}$$

This ends the proof of the Proposition.  $\square$



## └ Fractional dissipation case

Now, turn to prove Theorem 3. (For simplicity, we just give a *a priori* estimate)

First, we will give some estimates for the linear transport-diffusion model

$$\begin{cases} \partial_t \theta + v \cdot \nabla \theta + |D|\theta = f \\ \theta|_{t=0} = \theta^0. \end{cases} \quad (\text{TD})$$

We can easily get the  $L^p$  estimate, for  $p \in [1, \infty]$

$$\|\theta(t)\|_{L^p} \leq \|\theta^0\|_{L^p} + \int_0^t \|f(\tau)\|_{L^p} d\tau. \quad (19)$$

For  $p \in [1, \infty)$  there exists a constant  $C$  such that

$$\sup_{q \in \mathbb{N}} 2^q \|\Delta_q \theta\|_{L_t^1 L^p} \leq C \|\theta^0\|_{L^p} + C \|\theta^0\|_{L^\infty} \|\omega\|_{L_t^1 L^p}, \quad (20)$$

for every smooth solution  $\theta$  of (TD) with  $f = 0$ .

Applying Riesz transform  $\mathcal{R}$  to the temperature equation we get

$$\partial_t \mathcal{R}\theta + v \cdot \nabla \mathcal{R}\theta + |D|\mathcal{R}\theta = -[\mathcal{R}, v \cdot \nabla]\theta. \quad (21)$$

Since  $|D|\mathcal{R} = \partial_1$  then the function  $\Gamma := \omega + \mathcal{R}\theta$  satisfies

$$\partial_t \Gamma + v \cdot \nabla \Gamma = -[\mathcal{R}, v \cdot \nabla]\theta. \quad (22)$$

According to the first part of Proposition 5

$$\|[\mathcal{R}, v \cdot \nabla]\theta\|_{B_{p,r}^0} \leq C \|\nabla v\|_{L^p} (\|\theta\|_{B_{\infty,r}^0} + \|\theta\|_{L^p}),$$

applied with  $r = 2$  we have

$$\|[\mathcal{R}, v \cdot \nabla]\theta\|_{B_{p,2}^0} \lesssim \|\nabla v\|_{L^p} (\|\theta\|_{B_{\infty,2}^0} + \|\theta\|_{L^p}).$$

## └ Fractional dissipation case

Using the classical embedding  $B_{p,2}^0 \hookrightarrow L^p$  which is true only for  $p \in [2, \infty)$

$$\|[\mathcal{R}, v \cdot \nabla]\theta\|_{L^p} \leq \|\nabla v\|_{L^p} (\|\theta\|_{B_{\infty,2}^0} + \|\theta\|_{L^p}).$$

Since  $\operatorname{div} v = 0$  then we get from the transport equation (22)

$$(\partial_t \Gamma + v \cdot \nabla \Gamma = -[\mathcal{R}, v \cdot \nabla]\theta)$$

$$\|\Gamma(t)\|_{L^p} \leq \|\Gamma^0\|_{L^p} + \int_0^t \|[\mathcal{R}, v \cdot \nabla]\theta(\tau)\|_{L^p} d\tau.$$

Putting together the last two estimates we get

$$\begin{aligned} \|\Gamma(t)\|_{L^p} &\lesssim \|\Gamma^0\|_{L^p} + \int_0^t \|\nabla v(\tau)\|_{L^p} (\|\theta(\tau)\|_{B_{\infty,2}^0} + \|\theta\|_{L^p}) d\tau \\ &\lesssim \|\omega^0\|_{L^p} + \|\theta^0\|_{L^p} + \int_0^t \|\omega(\tau)\|_{L^p} (\|\theta(\tau)\|_{B_{\infty,2}^0} + \|\theta^0\|_{L^p}) d\tau. \end{aligned}$$

## └ Fractional dissipation case

On the other hand, from the continuity of the Riesz transform  
 Proposition 4 ( $\|\mathcal{R}\|_{\mathcal{L}(L^p)} \lesssim 1$ ) and (19) ( $\|\theta(t)\|_{L^p} \leq \|\theta^0\|_{L^p} + \int_0^t \|f(\tau)\|_{L^p} d\tau$ )

$$\begin{aligned}\|\omega(t)\|_{L^p} &\leq \|\Gamma(t)\|_{L^p} + \|\mathcal{R}\theta\|_{L^p} \\ &\lesssim \|\Gamma(t)\|_{L^p} + \|\theta^0\|_{L^p}.\end{aligned}$$

This leads to

$$\|\omega(t)\|_{L^p} \lesssim \|\omega^0\|_{L^p} + \|\theta^0\|_{L^p} + \int_0^t \|\omega(\tau)\|_{L^p} (\|\theta(\tau)\|_{B_{\infty,2}^0} + \|\theta^0\|_{L^p}) d\tau.$$

According to Gronwall lemma we get

$$\|\omega(t)\|_{L^p} \leq C_0 e^{C_0 t} e^{C\|\theta\|_{L_t^1 B_{\infty,2}^0}}. \quad (23)$$

## └ Fractional dissipation case

Let  $N \in \mathbb{N}$ , by Bernstein inequalities and (19)

$$\begin{aligned}
 \|\theta\|_{L_t^1 B_{\infty,2}^0} &\leq \|S_N \theta\|_{L_t^1 B_{\infty,2}^0} + \|(\text{Id} - S_N)\theta\|_{L_t^1 B_{\infty,1}^0} \\
 &\lesssim t \|\theta^0\|_{L^\infty} \sqrt{N} + \sum_{q \geq N} \|\Delta_q \theta\|_{L_t^1 L^\infty} \\
 &\lesssim \sqrt{N} \|\theta^0\|_{L^\infty} t + \sum_{q \geq N} 2^{q \frac{2}{p}} \|\Delta_q \theta\|_{L_t^1 L^p}.
 \end{aligned}$$

Using (20) ( $\sup_{q \in \mathbb{N}} 2^q \|\Delta_q \theta\|_{L_t^1 L^p} \leq C \|\theta^0\|_{L^p} + C \|\theta^0\|_{L^\infty} \|\omega\|_{L_t^1 L^p}$ ) and  $p > 2$  we obtain

$$\begin{aligned}
 \sum_{q \geq N-1} 2^{q \frac{2}{p}} \|\Delta_q \theta\|_{L_t^1 L^p} &\lesssim \sum_{q \geq N-1} 2^{q(\frac{2}{p}-1)} \left( \|\theta^0\|_{L^p} + \|\theta^0\|_{L^\infty} \int_0^t \|\omega(\tau)\|_{L^p} d\tau \right) \\
 &\lesssim \|\theta^0\|_{L^p} + 2^{N(-1+\frac{2}{p})} \|\theta^0\|_{L^\infty} \int_0^t \|\omega(\tau)\|_{L^p} d\tau.
 \end{aligned}$$

## └ Fractional dissipation case

Thus, we get

$$\|\theta\|_{L_t^1 B_{\infty,2}^0} \lesssim \sqrt{N} \|\theta^0\|_{L^\infty} t + \|\theta^0\|_{L^p} + 2^{N(-1+\frac{2}{p})} \|\theta^0\|_{L^\infty} \int_0^t \|\omega(\tau)\|_{L^p} d\tau.$$

We choose  $N$  as follows

$$N = \left\lceil \frac{\log \left( e + \int_0^t \|\omega(\tau)\|_{L^p} d\tau \right)}{(1 - 2/p) \log 2} \right\rceil + 1.$$

Then it follows

$$\|\theta\|_{L_t^1 B_{\infty,2}^0} \lesssim \|\theta^0\|_{L^\infty \cap L^p} + \|\theta^0\|_{L^\infty} t \log^{\frac{1}{2}} \left( e + \int_0^t \|\omega(\tau)\|_{L^p} d\tau \right).$$

Combining this estimate with (23) ( $\|\omega(t)\|_{L^p} \leq C_0 e^{C_0 t} e^{C\|\theta\|_{L_t^1 B_{\infty,2}^0}}$ ) we get

$$\begin{aligned} \|\theta\|_{L_t^1 B_{\infty,2}^0} &\lesssim \|\theta^0\|_{L^\infty \cap L^p} + \|\theta^0\|_{L^\infty} t \log^{\frac{1}{2}} \left( e + C_0 e^{C_0 t} e^{C\|\theta\|_{L_t^1 B_{\infty,2}^0}} \right) \\ &\leq C_0(1 + t^2) + C\|\theta^0\|_{L^\infty} t \|\theta\|_{L_t^1 B_{\infty,2}^0}^{\frac{1}{2}}. \end{aligned}$$

Thus we get for every  $t \in \mathbb{R}_+$

$$\|\theta\|_{L_t^1 B_{\infty,2}^0} \leq C_0(1 + t^2).$$

## └ Fractional dissipation case

It follows from (23)  $(\|\omega(t)\|_{L^p} \leq C_0 e^{C_0 t} e^{C\|\theta\|_{L_t^1 B_{\infty,2}^0}})$

$$\|\omega(t)\|_{L^p} \leq \Phi_1(t). \quad (24)$$

Applying (20)  $(\sup_{q \in \mathbb{N}} 2^q \|\Delta_q \theta\|_{L_t^1 L^p} \leq C \|\theta^0\|_{L^p} + C \|\theta^0\|_{L^\infty} \|\omega\|_{L_t^1 L^p})$  and (24) we get

$$2^q \|\Delta_q \theta\|_{L_t^1 L^p} \leq \Phi_1(t), \quad \forall q \in \mathbb{N} \quad (25)$$

and thus

$$\|\theta\|_{\tilde{L}_t^1 B_{p,\infty}^1} \leq \Phi_1(t).$$

Combing with (24), we have

$$\|\omega(t)\|_{L^p} + \|\theta\|_{\tilde{L}_t^1 B_{p,\infty}^1} \leq \Phi_1(t), \quad p \in (2, \infty). \quad (26)$$



It is not hard to see that from (25) ( $2^q \|\Delta_q \theta\|_{L_t^1 L^p} \leq \Phi_1(t)$ ) one can obtain that for every  $s < 1$

$$\|\theta\|_{L_t^1 B_{p,1}^s} \leq \|\theta\|_{\tilde{L}_t^1 B_{p,\infty}^1} \leq \Phi_1(t). \quad (27)$$

Combined with Bernstein inequalities and the fact that  $p > 2$  this yields

$$\|\theta\|_{L_t^1 B_{\infty,1}^\epsilon} \leq \Phi_1(t), \quad (28)$$

for every  $\epsilon < 1 - \frac{2}{p}$ .

By using the maximum principle for the transport equation (22)

$(\partial_t \Gamma + v \cdot \nabla \Gamma = -[\mathcal{R}, v \cdot \nabla] \theta)$  , we get

$$\|\Gamma(t)\|_{L^\infty} \leq \|\Gamma^0\|_{L^\infty} + \int_0^t \|[\mathcal{R}, v \cdot \nabla] \theta(\tau)\|_{L^\infty} d\tau.$$

Since the function  $\mathcal{R}\theta$  satisfies the equation

$$(\partial_t + v \cdot \nabla + |D|) \mathcal{R}\theta = -[\mathcal{R}, v \cdot \nabla] \theta, \quad (29)$$

we get by using (19)  $(\|\theta(t)\|_{L^p} \leq \|\theta^0\|_{L^p} + \int_0^t \|f(\tau)\|_{L^p} d\tau)$  for  $p = \infty$  that

$$\|\mathcal{R}\theta(t)\|_{L^\infty} \leq \|\mathcal{R}\theta^0\|_{L^\infty} + \int_0^t \|[\mathcal{R}, v \cdot \nabla] \theta(\tau)\|_{L^\infty} d\tau.$$

Combining the last two estimates yields

$$\begin{aligned}\|\Gamma(t)\|_{L^\infty} + \|\mathcal{R}\theta(t)\|_{L^\infty} &\leq \|\Gamma^0\|_{L^\infty} + \|\mathcal{R}\theta^0\|_{L^\infty} + 2 \int_0^t \|[\mathcal{R}, \nu \cdot \nabla]\theta(\tau)\|_{L^\infty} d\tau \\ &\leq C_0 + \int_0^t \|[\mathcal{R}, \nu \cdot \nabla]\theta(\tau)\|_{B_{\infty,1}^0} d\tau.\end{aligned}$$

It follows from the second estimate of Proposition 5 ( $\|[\mathcal{R}, \nu \cdot \nabla]\theta\|_{B_{\infty,r}^0} \leq C(\|\omega\|_{L^\infty} + \|\omega\|_{L^p})(\|\theta\|_{B_{\infty,r}^\epsilon} + \|\theta\|_{L^p})$ ) and (26) ( $\|\omega(t)\|_{L^p} + \|\theta\|_{\tilde{L}_t^1 B_{p,\infty}^1} \leq \Phi_1(t)$ )

$$\begin{aligned}\|\omega(t)\|_{L^\infty} + \|\mathcal{R}\theta(t)\|_{L^\infty} &\lesssim C_0 + \int_0^t \|\omega(\tau)\|_{L^\infty \cap L^p} (\|\theta(\tau)\|_{B_{\infty,1}^\epsilon} + \|\theta(\tau)\|_{L^p}) d\tau \\ &\lesssim C_0 + \|\omega\|_{L_t^\infty L^p} (\|\theta\|_{L_t^1 B_{\infty,1}^\epsilon} + t \|\theta^0\|_{L^p}) \\ &\quad + \int_0^t \|\omega(\tau)\|_{L^\infty} (\|\theta(\tau)\|_{B_{\infty,1}^\epsilon} + \|\theta^0\|_{L^p}) d\tau.\end{aligned}$$

## └ Fractional dissipation case

Let  $0 < \epsilon < 1 - \frac{2}{p}$  then using (28) ( $\|\theta\|_{L_t^1 B_{\infty,1}^\epsilon} \leq \Phi_1(t)$ ) we get

$$\|\omega(t)\|_{L^\infty} + \|\mathcal{R}\theta(t)\|_{L^\infty} \lesssim \Phi_1(t) + \int_0^t \|\omega(\tau)\|_{L^\infty} (\|\theta(\tau)\|_{B_{\infty,1}^\epsilon} + \|\theta^0\|_{L^p}) d\tau.$$

Therefore we obtain by the Gronwall lemma and a new use of (28) that

$$\|\omega(t)\|_{L^\infty} + \|\mathcal{R}\theta(t)\|_{L^\infty} \leq \Phi_2(t). \quad (30)$$

Let  $N \in \mathbb{N}$  to be chosen later. Using the fact that  $\|\dot{\Delta}_q v\|_{L^\infty} \approx 2^{-q} \|\dot{\Delta}_q \omega\|_{L^\infty}$ , we then have

$$\begin{aligned} \|v(t)\|_{L^\infty} &\leq \|\chi(2^{-N}|D|)v(t)\|_{L^\infty} + \sum_{q \geq -N} 2^{-q} \|\dot{\Delta}_q \omega(t)\|_{L^\infty} \\ &\leq \|\chi(2^{-N}|D|)v(t)\|_{L^\infty} + 2^N \|\omega(t)\|_{L^\infty}. \end{aligned}$$

## └ Fractional dissipation case

Applying the frequency localizing operator to the velocity equation we get

$$\begin{aligned}\chi(2^{-N}|D|)v &= \chi(2^{-N}|D|)v_0 + \int_0^t \mathcal{P}\chi(2^{-N}|D|)\theta(\tau)d\tau \\ &\quad + \int_0^t \mathcal{P}\chi(2^{-N}|D|)\operatorname{div}(v \otimes v)(\tau)d\tau.\end{aligned}$$

where  $\mathcal{P}$  stands for Leray projector. From Bernstein inequalities, Calderón-Zygmund estimate and the uniform boundness of  $\chi(2^{-N}|D|)$  we get

$$\begin{aligned}\int_0^t \|\chi(2^{-N}|D|)\mathcal{P}\theta(\tau)\|_{L^\infty}d\tau &\lesssim 2^{-N\frac{2}{p}} \int_0^t \|\theta(\tau)\|_{L^p}d\tau \\ &\lesssim t\|\theta^0\|_{L^p}.\end{aligned}$$

## └ Fractional dissipation case

Using Proposition 4-(ii) ( $\| |D|^s \chi(2^{-q}|D|) \mathcal{R} \|_{\mathcal{L}(L^p)} \leq C 2^{qs}$ ) we find

$$\int_0^t \| \mathcal{P} \chi(2^{-N}|D|) \operatorname{div}(v \otimes v)(\tau) \|_{L^\infty} d\tau \lesssim 2^{-N} \int_0^t \| v(\tau) \|_{L^\infty}^2 d\tau.$$

The outcome is

$$\begin{aligned} \| v(t) \|_{L^\infty} &\lesssim \| v^0 \|_{L^\infty} + t \| \theta_0 \|_{L^p} + 2^{-N} \int_0^t \| v(\tau) \|_{L^\infty}^2 d\tau + 2^N \| \omega(t) \|_{L^\infty} \\ &\lesssim 2^{-N} \int_0^t \| v(\tau) \|_{L^\infty}^2 d\tau + 2^N \Phi_2(t) \end{aligned}$$

Choosing judiciously  $N$  we find

$$\| v(t) \|_{L^\infty} \leq \Phi_2(t) \left( 1 + \left( \int_0^t \| v(\tau) \|_{L^\infty}^2 d\tau \right)^{\frac{1}{2}} \right).$$

From Gronwall lemma we get

$$\|v(t)\|_{L^\infty} \leq \Phi_3(t). \quad (31)$$

First, we need a logarithmic estimates of the transport equation

$$(\partial_t + v \cdot \nabla + \kappa|D|)\theta = f,$$

then we have

$$\|\theta\|_{\tilde{L}_t^\infty B_{p,1}^0} \leq C \left( \|\theta^0\|_{B_{p,1}^0} + \|f\|_{L_t^1 B_{p,1}^0} \right) \left( 1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \right), \quad (32)$$

for  $p \in [1, \infty]$  (Details see Theorem 4.5 in [Hmidi, Keraani and Rousset](#) 2011).

By using the logarithmic estimates of the equations (22)  $(\partial_t \Gamma + v \cdot \nabla \Gamma = -[\mathcal{R}, v \cdot \nabla] \theta)$  and (29)  $((\partial_t + v \cdot \nabla + |D|) \mathcal{R} \theta = -[\mathcal{R}, v \cdot \nabla] \theta)$ , we obtain

$$\|\Gamma(t)\|_{B_{\infty,1}^0} + \|\mathcal{R}\theta(t)\|_{B_{\infty,1}^0} \lesssim \left( C_0 + \|[\mathcal{R}, v \cdot \nabla] \theta\|_{L_t^1 B_{\infty,1}^0} \right) \left( 1 + \|\nabla v\|_{L_t^1 L^\infty} \right). \quad (33)$$

Thanks to Propositions 5  $(\|[\mathcal{R}, v \cdot \nabla] \theta\|_{B_{\infty,r}^0} \leq C(\|\omega\|_{L^\infty} + \|\omega\|_{L^p})(\|\theta\|_{B_{\infty,r}^\epsilon} + \|\theta\|_{L^p}))$ , (26)  $(\|\omega(t)\|_{L^p} + \|\theta\|_{\tilde{L}_t^1 B_{p,\infty}^1} \leq \Phi_1(t))$  (30)  $(\|\omega(t)\|_{L^\infty} + \|\mathcal{R}\theta(t)\|_{L^\infty} \leq \Phi_2(t))$  and (28)  $(\|\theta\|_{L_t^1 B_{\infty,1}^\epsilon} \leq \Phi_1(t))$  we get

$$\begin{aligned} \|[\mathcal{R}, v \cdot \nabla] \theta\|_{L_t^1 B_{\infty,1}^0} &\lesssim \int_0^t (\|\omega(\tau)\|_{L^\infty} + \|\omega(\tau)\|_{L^p}) \\ &\quad (\|\theta(\tau)\|_{B_{\infty,1}^\epsilon} + \|\theta(\tau)\|_{L^p}) d\tau \\ &\lesssim \Phi_2(t). \end{aligned}$$



## └ Fractional dissipation case

By easy computations we get

$$\begin{aligned}
 \|\nabla v\|_{L^\infty} &\leq \|\nabla \Delta_{-1} v\|_{L^\infty} + \sum_{q \in \mathbb{N}} \|\Delta_q \nabla v\|_{L^\infty} \\
 &\lesssim \|\omega\|_{L^p} + \sum_{q \in \mathbb{N}} \|\Delta_q \omega\|_{L^\infty} \\
 &\lesssim \Phi_1(t) + \|\omega(t)\|_{B_{\infty,1}^0}.
 \end{aligned} \tag{34}$$

Putting together (33) and (34) leads to

$$\begin{aligned}
 \|\omega(t)\|_{B_{\infty,1}^0} &\leq \|\Gamma(t)\|_{B_{\infty,1}^0} + \|\mathcal{R}\theta(t)\|_{B_{\infty,1}^0} \\
 &\leq \Phi_2(t) \left( 1 + \int_0^t \|\omega(\tau)\|_{B_{\infty,1}^0} d\tau \right).
 \end{aligned}$$

Thus we obtain from Gronwall inequality

$$\|\omega(t)\|_{B_{\infty,1}^0} \leq \Phi_3(t). \tag{35}$$

Coming back to (34) ( $\|\nabla v\|_{L^\infty} \lesssim \Phi_1(t) + \|\omega(t)\|_{B_{\infty,1}^0}$ ) we get

$$\|\nabla v(t)\|_{L^\infty} \leq \Phi_3(t).$$

Let us move to the estimate of  $v$  in the space  $B_{\infty,1}^1$ . By definition we have

$$\|v(t)\|_{B_{\infty,1}^1} \lesssim \|v(t)\|_{L^\infty} + \|\omega(t)\|_{B_{\infty,1}^0}.$$

Combined with (31) ( $\|v(t)\|_{L^\infty} \leq \Phi_3(t)$ ) and (35) ( $\|\omega(t)\|_{B_{\infty,1}^0} \leq \Phi_3(t)$ ) this yields

$$\|v(t)\|_{B_{\infty,1}^1} \leq \Phi_3(t).$$

# General critical case

Jiu, Miao Wu and Zhang obtained the global regularity for the general critical dissipation  $\alpha + \beta = 1$ .

Jiu, Quansen; Miao, Changxing; Wu, Jiahong; Zhang, Zhifei The two-dimensional incompressible Boussinesq equations with general critical dissipation. SIAM J. Math. Anal. 46 (2014).

## Theorem 7 (Jiu, Miao, Wu, Zhang, 2014)

Let  $\alpha_0 < \alpha < 1$  and  $\alpha + \beta = 1$ , where

$$\alpha_0 = \frac{23 - \sqrt{145}}{12} \approx 0.9132. \quad (36)$$

Assume that  $u_0 \in B_{2,1}^\sigma(\mathbb{R}^2)$  with  $\sigma \geq \frac{5}{2}$  and  $\theta_0 \in B_{2,1}^2(\mathbb{R}^2)$ . Then the fractional Boussinesq has a unique global solution  $(u, \theta)$  satisfying, for any  $0 < T < \infty$ ,

$$\begin{aligned} u &\in C([0, T]; B_{2,1}^\sigma(\mathbb{R}^2)) \cap L^1([0, T]; B_{2,1}^{\sigma+\alpha}(\mathbb{R}^2)), \\ \theta &\in C([0, T]; B_{2,1}^2(\mathbb{R}^2)) \cap L^1([0, T]; B_{2,1}^{2+\beta}(\mathbb{R}^2)). \end{aligned} \quad (37)$$

The authors also introduced the new quantity

$$G = \omega - \mathcal{R}_\alpha \theta \quad \text{with} \quad \mathcal{R}_\alpha = \Lambda^{-\alpha} \partial_1,$$

which satisfies

$$\partial_t G + u \cdot \nabla G + \Lambda^\alpha G = [\mathcal{R}_\alpha, u \cdot \nabla] \theta + \Lambda^{\beta-\alpha} \partial_1 \theta.$$

The regularity of  $G$  here does not translate to the regularity on the vorticity  $\omega$  in the general case, since the corresponding regularity of  $\mathcal{R}_\alpha \theta$  is not known for  $\alpha < 1$ .

By obtaining a suitable estimate for  $[\mathcal{R}_\alpha, u \cdot \nabla]\theta$ , they obtained a global bound for  $\|G\|_{L^2}$  when  $\alpha > \frac{4}{5}$ . In addition, with  $\alpha > \frac{4}{5}$ , a global bound is also established for  $\|G\|_{L^q}$  when  $q$  is in the range

$$2 < q < q_0 = \frac{8 - 4\alpha}{8 - 7\alpha}.$$

This global bound for  $\|G\|_{L^q}$  enables them to estimate  $\|G\|_{B_{q,\infty}^s}$ ,

$$\|G(t)\|_{B_{q,\infty}^s} \leq C,$$

for  $\alpha_0 < \alpha$  and  $s \leq 3\alpha - 2$ , and this will give a bound for part of  $\nabla u$ .

Through the temperature equation

$$\partial_t \theta + u \cdot \nabla \theta + \Lambda^\beta \theta = 0. \quad (38)$$

They offered a different approach to gaining further regularity by  $\theta$ . Since  $u$  is determined by  $\omega$  through  $u = \nabla^\perp \Delta^{-1} \omega$  and  $\omega = G + \mathcal{R}_\alpha \theta$ , So  $u$  can be decomposed into two parts,

$$u = \nabla^\perp \Delta^{-1} \omega = \nabla^\perp \Delta^{-1} G + \nabla^\perp \Delta^{-1} \mathcal{R}_\alpha \theta \equiv \tilde{u} + v.$$

and  $\tilde{u}$  is more regular in the sense that

$$\|\nabla \tilde{u}\|_{L^\infty} = \|\nabla \nabla^\perp \Delta^{-1} G\|_{L^\infty} \leq C \|G\|_{B_{q,\infty}^s} \leq C.$$

In addition, when  $\alpha + \beta = 1$ ,  $v$  in terms of  $\theta$  can be written as

$$v = \nabla^\perp \Delta^{-1} \Lambda^{-(1-\beta)} \partial_1 \theta.$$

Therefore, (38) is almost a generalized critical surface quasi-geostrophic (SQG) type equation.



# Subcritical case

P. Constantin and V. Vicol, GFA, 2012. Global regularity for the subcritical case by a different approach.

Theorem 8 (P. Constantin and V. Vicol 2012)

*Assume that  $(u_0, \theta_0) \in \mathcal{S}$ , the Schwartz class. If  $\beta > \frac{2}{2+\alpha}$ , then the Boussinesq equations has a unique global smooth solution.*

The proof involves the pointwise inequality for fractional Laplacian

$$\nabla f \cdot \Lambda^\alpha \nabla f(x) \geq \frac{1}{2} \Lambda^\alpha |\nabla f|^2 + \frac{|\nabla f(x)|^{2+\alpha}}{c \|f\|_{L^\infty}^\alpha}.$$

Note that in this theorem  $\alpha + \beta > 1$ , a subcritical case.