

# An extension of Calderón-Zygmund type singular integral

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December 4, 2020

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# Contents

- 1 The classical CZ singular integral
- 2 An extension of CZ singular integral
- 3 An application in SQG

# 1. The classical CZ singular integral

The classical Calderón-Zygmund type singular integral is defined as

$$I_\varepsilon(f)(x) = \int_{|x-y| \geq \varepsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy$$

for any  $\varepsilon > 0$  and  $f \in L^q(\mathbb{R}^n)$ ,  $1 \leq q < \infty$ , where  $\Omega(x)$  satisfies, for any  $r > 0$ ,

$$\left\{ \begin{array}{ll} |\Omega(x)| \leq B_1, & \text{(bounded)} \\ \Omega(rx) = \Omega(x), & \text{(homogeneous of degree 0)} \\ \int_{\mathbb{S}^1} \Omega(x) d\sigma = 0, & \text{(cancellation)} \\ \int_0^1 \frac{\omega(\delta)}{\delta} d\delta = B_2, & \text{(Dini type continuous)} \end{array} \right. \quad (I)$$

where  $\mathbb{S}^1 = \{x \in \mathbb{R}^n : |x| = 1\}$  is the unit sphere in  $\mathbb{R}^n$  and  $\omega$  is defined as

$$\omega(\delta) = \sup\{|\Omega(x) - \Omega(x')| : |x - x'| \leq \delta, |x| = |x'| = 1\}.$$

The well-known Calderón-Zygmund estimate reads as

### Theorem

Suppose  $1 < q < \infty$ . There exists a constant  $A$  depending only on  $n, q, B_1$  and  $B_2$  such that  $\|I_\varepsilon f\|_q \leq A\|f\|_q$  for any  $\varepsilon > 0$  and  $f \in L^q(\mathbb{R}^n)$ .

### Remark

The Calderón-Zygmund singular integral  $I_\varepsilon$  is of strong type  $(q, q)$  for  $1 < q < \infty$ .

Consider the Poisson equation

$$\Delta u = f(x), \quad x \in \mathbb{R}^n \quad (II)$$

for  $f \in C_0^\infty(\mathbb{R}^n)$

The solution of (II) is expressed as

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y)f(y)dy,$$

where

$$\Gamma(x-y) = \Gamma(|x-y|) = \begin{cases} \frac{1}{n(2-n)\omega_n} |x-y|^{2-n}, & n > 2, \\ \frac{1}{2\pi} \log |x-y|, & n = 2 \end{cases}$$

is the fundamental solution of the Laplacian equation.

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy = \lim_{\varepsilon \rightarrow 0} \int_{|x-y| > \varepsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy,$$

where  $\Omega(x) = \frac{1}{n\omega_n}(\delta_{ij} - n|x|^{-2}x_i x_j)$  for  $i, j = 1, 2, \dots, n$ , which satisfy (I).

Another way to express the solution of (II) is as follows:

$$\begin{aligned} \left( \frac{\partial^2 u}{\partial x_j \partial x_k} \right) \mathfrak{r} &= -4\pi^2 \xi_j \xi_k \hat{u} \\ &= -\left( \frac{i\xi_j}{|\xi|} \right) \left( \frac{i\xi_k}{|\xi|} \right) (-4\pi |\xi|^2) \hat{u} = -(R_j R_k \Delta u) \mathfrak{r}, \end{aligned}$$

where

$$\hat{f}(y) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot y} f(x) dx,$$

and  $R_j, R_k$  are Riesz transformation defined by

$$\widehat{R_j f}(\xi) = i \frac{\xi_j}{|\xi|} \hat{f}(\xi), \quad j = 1, 2, \dots, n,$$



Or

$$R_j f(x) = c_n \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} f(y) dy = c_n \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy,$$

where  $c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}$ ,  $\Omega(x) = \frac{x_j}{|x|}$  satisfy (I). Therefore

$$\frac{\partial^2 u}{\partial x_j \partial x_k} = -R_j R_k \Delta u = -R_j R_k f.$$

### Remark

In particular, the Riesz transform is of strong type  $(q, q)$ :  $\|R_j f\|_q \leq C \|f\|_q$  for  $1 < q < \infty$ .

### Remark

The solution to (II) satisfies  $\|D^2 u\|_q \leq C \|f\|_q$  for  $1 < q < \infty$ .

# Ingredients of Proof of the Calderón-Zygmund estimate

Method 1( Calderón-Zygmund, Acta Math (1952); Stein(1970)). The original proof contains four steps:

- prove  $I_\varepsilon$  is of strong type  $(2, 2)$  via Fourier transformation;
- prove  $I_\varepsilon$  is of weak type  $(1, 1)$  via Calderón-Zygmund decomposition;

$$|\{x \in \mathbb{R}^n : |I_\varepsilon f(x)| > \lambda\}| \leq C \frac{\|f\|_1}{\lambda}$$

for any  $\lambda > 0$ .

- prove  $I_\varepsilon$  is of strong type  $(p, p)$  for  $1 < p < 2$  via Marcinkiewicz's interpolation;
- prove  $I_\varepsilon$  is of strong type  $(p, p)$  for  $2 < p < \infty$  via duality method.

# Ingredients of Proof of the Calderón-Zygmund estimate

Method 2( Caffarelli-Paral, CPAM(1998);Li-Wang, Arch. Math.(2006))

"Geometric approach" contain three steps

- prove  $I_\varepsilon$  is of strong type  $(2, 2)$  via Fourier transformation;
- prove  $I_\varepsilon$  is of strong type  $(p, p)$  for  $2 < p < \infty$ ;
- prove  $I_\varepsilon$  is of strong type  $(p, p)$  for  $1 < p < 2$  via duality method.

## Geometric approach

$$\begin{aligned}\|f\|_p^p &= p \int_0^\infty t^{p-1} |\{x : |f(x)| > t\}| dt \\&= p \sum_{k=-\infty}^\infty \int_{\lambda N^{k-1}}^{\lambda N^k} t^{p-1} |\{x : |f(x)| > t\}| dt \\&\leq p \sum_{k=-\infty}^\infty |\{x : |f(x)| > \lambda N^{k-1}\}| \int_{\lambda N^{k-1}}^{\lambda N^k} t^{p-1} dt \\&\leq \sum_{k=-\infty}^\infty |\{x : |f(x)| > \lambda N^{k-1}\}| (\lambda N^k)^p\end{aligned}$$

for any  $\lambda > 0, N > 1$ . Then when  $f \in L^2$  and  $p > 2$ , only need to prove that the distribution function  $|\{x : |f(x)| > \lambda N^{k-1}\}|$  decays fast as  $k$  becomes large, for example,  $|\{x : |f(x)| > \lambda N^{k-1}\}| \leq N^{-k(p+\delta)}$  for any  $\delta > 0$ .

## 2. An extension of CZ singular integral

Now we consider the following singular integral

$$T_\varepsilon(f)(x) = \int_{|y| \geq \varepsilon} \frac{\Omega(y)}{|y|^{n-\beta}} f(x-y) dy \quad (*)$$

for any  $\varepsilon > 0$  and  $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ ,  $1 \leq q < \infty$  and  $0 < \beta < n$ , where  $\Omega$  is same as above. Formally,  $T_\varepsilon$  becomes the Calderón-Zygmund type singular integral  $I_\varepsilon$  when  $\beta = 0$ .

# Main Result

Our main result can be stated as

## Theorem 1 (Jiu-Li-Yu, 2020, to appear in JFA)

Let  $0 < \beta_0 < \frac{1}{2}$  be any fixed and small number. Then for any  $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$  with  $1 < q < \infty$ , there exists an absolute constant  $C$  depending on  $n, q, B_1, B_2$  and  $\beta_0$  such that

$$\|T_\varepsilon f\|_q \leq C \left( \|f\|_q + \frac{\beta^{\frac{(q-1)n}{q}}}{\sqrt[q]{n(q-1) - \beta q}} \|f\|_1 \right) \quad (1)$$

holds uniformly for  $\varepsilon > 0$  and  $0 < \beta < \min\{1 - \beta_0, \frac{(q-1)n}{q}\}$ .

# Remarks

## Riesz Potential

Riesz potential is

$$I_{\beta}f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta}} dy, 0 < \beta < n.$$

## Hardy-Littlewood-Sobolev inequality

Let  $0 < \beta < n$ ,  $1 < p < q < \infty$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$ . Then,

$$\|I_{\beta}f\|_{L^q} \leq A_{p,q,\beta} \|f\|_{L^p},$$

## Remarks

In view of the Hardy-Littlewood-Sobolev inequality , it is direct to obtain

$$\|T_\varepsilon f\|_q \leq A_{p,q,\beta} \|f\|_p, \quad \frac{1}{q} = \frac{1}{p} - \frac{\beta}{n},$$

However, the constant  $A_{p,q,\beta}$  depends on  $\beta$ . When  $\beta \rightarrow 0$ ,  $A_{p,q,\beta} \rightarrow \infty$ .

The constant  $C$  on the right hand of (1) does not depend on  $\beta$  and the strong  $(q, q)$  type estimate of the Calderón-Zygmund type singular integral  $I_\varepsilon$  can be recovered from (\*) when  $\beta \rightarrow 0$ . In this sense, the singular integral  $T_\varepsilon$  in (\*) can be viewed as an extension of the Calderón-Zygmund type singular integral.



## An extension of Riesz transform

If  $\Omega(x) = \frac{x_j}{|x|}$  for  $j = 1, 2, \dots, n$  respectively, we have proved

**Theorem 2 (H. Yu, Q. Jiu, Nonlinear Analysis: RWA 2019)**

Take  $\Omega(x) = \frac{x_j}{|x|}$  for  $j = 1, 2, \dots, n$  in (\*) respectively. Let  $0 < \beta_0 < \frac{1}{2}$  be any fixed and small number. Then for any  $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$  with  $1 < q < \infty$ , there exists an absolute constant  $C$  depending on  $n, q$  and  $\beta_0$  such that

(1) if  $q = 2$ , there holds

$$\|T_\varepsilon f\|_2 \leq C \left( \|f\|_2 + \frac{\beta^{\frac{n}{2}}}{\sqrt{n-2\beta}} \|f\|_1 \right)$$

for  $0 < \beta < 1 - \beta_0$ ;

# An extension of Riesz transform

Theorem 2 (H. Yu, Q. Jiu, Nonlinear Analysis: RWA 2019)

(2) if  $1 < q < 2$ , there holds

$$\begin{aligned} \|T_\varepsilon f\|_q \leq C & \left( \frac{1}{\sqrt[q]{(q-1)(2-q)}} \|f\|_q + \frac{1}{\sqrt[q]{q(n-\beta)^2 - n^2}} \|f\|_p \right. \\ & \left. + \frac{\beta^{\frac{n(q-1)}{q}}}{\sqrt[q]{(n(q-1) - \beta q)}} \|f\|_1 \right), \end{aligned}$$

where  $\frac{1}{q} = \frac{1}{p}(1 - \frac{\beta}{n})$ ,  $0 < \beta < \min\{1 - \beta_0, \frac{(q-\sqrt{q})n}{q}\}$ ;

# An extension of Riesz transform

Theorem 2 (H. Yu, Q. Jiu, Nonlinear Analysis: RWA 2019)

(3) if  $2 < q < \infty$ , there holds

$$\begin{aligned} \|T_\varepsilon f\|_{(L^{q'} \cap L^{p'})^*} &\leq C \max\left\{ \frac{1}{\sqrt[q']{(q' - 1)(2 - q')}}, \frac{1}{\sqrt[q']{q'(n - \beta)^2 - n^2}} \right\} \|f\|_q \\ &\quad + C \frac{\beta^{\frac{n(q-1)}{q}}}{\sqrt[q]{(n(q-1) - \beta q)}} \|f\|_1, \end{aligned}$$

where  $0 < \beta < \min\{1 - \beta_0, \frac{(q' - \sqrt{q'})n}{q'}\}$ ,  $\frac{1}{q} + \frac{1}{q'} = 1$ ,  $\frac{1}{q'} = \frac{1}{p'}(1 - \frac{\beta}{n})$ , and  $(L^{q'} \cap L^{p'})^*$  is the dual space of  $L^{q'} \cap L^{p'}$ .

## Remark

Taking  $\Omega(x) = \frac{x_j}{|x|}$  for  $j = 1, 2, \dots, n$  in  $(*)$ , it follows from Theorem 2 that the singular integral  $T_\varepsilon$  can be viewed as an extension of the Riesz transform. This kind of singular integral appears in the generalized surface quasi-geostrophic (SQG) equation.

## Remark

Theorem 1 extends Theorem 2, and the estimate in Theorem 1 is much more clean.

## Sketch of Proof

Let  $\chi(s) \in C_0^\infty(R)$  be the usual smooth cutting-off function which is defined as

$$\chi(s) = \begin{cases} 1, & |s| \leq 1, \\ 0, & |s| \geq 2, \end{cases}$$

satisfying  $|\chi'(s)| \leq 2$ . Let

$$\chi_\lambda(s) = \chi(\lambda s),$$

and define

$$T_1 f(x) = \int_{|y| \geq \varepsilon} \frac{\Omega(y)}{|y|^{n-\beta}} \chi_\beta(|y|) f(x-y) dy$$

$$T_2 f(x) = \int_{|y| \geq \varepsilon} \frac{\Omega(y)}{|y|^{n-\beta}} (1 - \chi_\beta(|y|)) f(x-y) dy.$$

Then  $T_\varepsilon$  in (\*), denoted by  $T$ , can be written as

$$T = T_1 + T_2.$$

# Sketch of Proof

## Lemma

There exists an absolute constant  $C$  depending on  $B_1$  and independent of  $\beta$  such that for any  $1 < q < \infty$ ,

$$\|T_2 f\|_q \leq C \frac{\beta^{\frac{(q-1)n}{q}}}{\sqrt[q]{(n(q-1) - \beta q)}} \|f\|_1, 0 < \beta < \frac{(q-1)n}{q}.$$

# Sketch of Proof

## Proof of Theorem 2. Method 1.

- prove  $T_1$  is of strong type  $(2, 2)$  via Fourier transformation;
- prove  $T_1$  is of weak type  $(1, 1)$  via Calderón-Zygmund decomposition;
- prove  $T_1$  is of strong type  $(p, p)$  for  $1 < p < 2$  via Marcinkiewicz's interpolation;
- prove  $T_1$  is of strong type  $(p, p)$  for  $2 < p < \infty$  via duality method.

# Sketch of Proof

## Proof of Theorem 1. Method 2. "Geometric Approach"

- prove  $T_1$  is of strong type  $(2, 2)$  via Fourier transformation;
- prove  $T_1$  is of strong type  $(p, p)$  for  $2 < p < \infty$ ;
- prove  $T_1$  is of strong type  $(p, p)$  for  $1 < p < 2$  via duality method.



# Sketch of Proof of Theorem 1

Lemma (Yu-Jiu (2019), Jiu-Li-Yu(2020))

$$\|T_1 f\|_2 \leq C \|f\|_2$$

holds for  $0 < \beta \leq 1 - \beta_0$ .

**Remark.** Denote

$$K_1(x) = \frac{\Omega(x)}{|x|^{n-\beta}} \chi_\beta(|x|).$$

It suffices to prove

$$\|\widehat{K_1}(y)\|_{L^\infty} \leq C, \quad 0 < \beta \leq 1 - \beta_0$$

**Remark.**  $T_1$  is of strong type (2,2).

# Sketch of Proof of Theorem 1

Next we prove that  $T_1$  is of strong type  $(q,q)$  for  $2 < q < \infty$ .

By "**Geometric Approach**", when  $q > 2$ , it suffices to prove that the distribution function  $|\{x : |T_1 f(x)| > \lambda N^k\}|$  decays fast as  $k$  becomes large, for and  $\lambda > 0$  and  $N > 1$ .

In application we use the maximal function  $\mathcal{M}f$  instead of  $f$ , which is defined as

$$\mathcal{M}f(x) = \sup\left\{\frac{1}{|Q|} \int_Q |f(y)| dy : Q \subset \mathbb{R}^n \text{ a cube, } x \in Q\right\}.$$

# Sketch of Proof of Theorem 1

## Lemma

It holds that

$$\|\mathcal{M}f\|_q \leq C_q \|f\|_q,$$

$$|\{x \in \mathbb{R}^n : (\mathcal{M}f)(x) > \lambda\}| \leq C_1 \frac{\|f\|_1}{\lambda}$$

for any  $\lambda > 0$ , where  $C_1$  is a positive constant depending only on  $n$  and  $C_q$  is a positive constant depending only on  $n$  and  $q$  for each  $1 < q < \infty$ .

# Sketch of Proof of Theorem 1

One of our key lemmas is

## Lemma 1 (Jiu-Li-Yu, 2020)

Let  $Q$  be a cube in  $\mathbb{R}^n$  and  $f \in L^2(\mathbb{R}^n)$  with  $\text{supp} f \subset \mathbb{R}^n \setminus 4Q$ . Suppose that  $\mathcal{M}(|T_1 f|^2)(x_0) \leq a^2$  and  $\mathcal{M}(f^2)(x_0) \leq b^2$  for a point  $x_0 \in 3Q$ , where  $a$  and  $b$  are two positive constants. Then, there exists an absolute constant  $C > 0$  depending on  $n, B_1$  and  $B_2$  but not on  $\beta$ , such that

$$\mathcal{M}(|T_1 f|^2)(x) \leq \max\{5^n a^2, (a + Cb)^2\} \quad (2)$$

holds for any  $x \in Q$  and  $0 < \beta < \frac{n}{2}$ .

# Sketch of Proof of Theorem 1

**Proof.** Claim

$$|T_1 f|(x) \leq a + Cb, \quad x \in 3Q.$$

In fact,

$$\begin{aligned} |T_1 f|(x) &\leq |T_1 f|(x_0) + |T_1 f(x) - T_1 f(x_0)| \\ &\leq \mathcal{M}(T_1 f)(x_0) + |T_1 f(x) - T_1 f(x_0)|. \end{aligned}$$

By Hölder inequality,

$$\mathcal{M}(T_1 f)(x_0) \leq \sqrt{\mathcal{M}(|T_1 f|^2)(x_0)} \leq a.$$

To prove the claim, we only need to prove

$$|T_1 f(x) - T_1 f(x_0)| \leq Cb. \quad (3)$$

Assume that **Claim** is true. We prove the Lemma, that is, the estimate (2) holds true. Fix  $x \in Q$  and let  $\hat{Q}$  be any cube containing  $x$ . If  $\hat{Q} \subseteq 3Q$ , then it follows from the **claim** that

$$\frac{1}{|\hat{Q}|} \int_{\hat{Q}} |T_1 f|^2(y) dy \leq (a + Cb)^2. \quad (4)$$

If  $\hat{Q} \not\subseteq 3Q$ , then  $x_0 \in 5\hat{Q}$  and in view of  $\mathcal{M}(|T_1 f|^2)(x_0) \leq a^2$ , we have

$$\frac{1}{|\hat{Q}|} \int_{\hat{Q}} |T_1 f|^2(y) dy \leq \frac{5^n}{|5\hat{Q}|} \int_{5\hat{Q}} |T_1 f|^2(y) dy \leq 5^n a^2. \quad (5)$$

Hence (2) follows from (4) and (5). The proof of the lemma is then complete.

Now we continue to prove the **Claim**. It suffices to prove (3), that is

$$|T_1 f(x) - T_1 f(x_0)| \leq Cb.$$

# Sketch of Proof of Theorem 1

Note that

$$\begin{aligned} & T_1 f(x) - T_1 f(x_0) \\ &= \int_{\mathbb{R}^n \setminus 4Q} \Omega(x-y) \left( \frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x_0-y|^{n-\beta}} \right) \chi_\beta(|x-y|) f(y) dy \\ &+ \int_{\mathbb{R}^n \setminus 4Q} \frac{\Omega(x-y) - \Omega(x_0-y)}{|x_0-y|^{n-\beta}} \chi_\beta(|x-y|) f(y) dy \\ &+ \int_{\mathbb{R}^n \setminus 4Q} \frac{\Omega(x_0-y)}{|x_0-y|^{n-\beta}} (\chi_\beta(|x-y|) - \chi_\beta(|x_0-y|)) f(y) dy \\ &\equiv I_1 + I_2 + I_3. \end{aligned} \tag{6}$$

# Sketch of Proof of Theorem 1

We first estimate  $I_1$ . It holds that

$$\begin{aligned} |I_1| &= \left| \int_{\mathbb{R}^n \setminus 4Q} \Omega(x-y) \left( \frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x_0-y|^{n-\beta}} \right) \chi_\beta(|x-y|) f(y) dy \right| \\ &\leq C \left( \int_{\mathbb{R}^n \setminus 4Q} \left( \frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x_0-y|^{n-\beta}} \right)^{\frac{n}{n-\beta}} |f(y)|^{\frac{n}{n-\beta}} dy \right)^{\frac{n-\beta}{n}} \\ &\quad \times \left( \int_{\mathbb{R}^n \setminus 4Q} |\chi_\beta(|x-y|)|^{\frac{n}{\beta}} dy \right)^{\frac{\beta}{n}} \\ &\leq C \left( \sum_{m=2}^{\infty} \int_{2^{m+1}Q \setminus 2^mQ} \left( \frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x_0-y|^{n-\beta}} \right)^{\frac{n}{n-\beta}} |f(y)|^{\frac{n}{n-\beta}} dy \right)^{\frac{n-\beta}{n}} \left[ \left( \frac{2}{\beta} \right)^n \right]^{\frac{\beta}{n}} \\ &\leq C \left( \sum_{m=2}^{\infty} \int_{2^{m+1}Q \setminus 2^mQ} \left( \frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x_0-y|^{n-\beta}} \right)^{\frac{n}{n-\beta}} |f(y)|^{\frac{n}{n-\beta}} dy \right)^{\frac{n-\beta}{n}}. \end{aligned} \tag{7}$$



# Sketch of Proof of Theorem 1

$$\begin{aligned} & \sum_{m=2}^{\infty} \int_{2^{m+1}Q \setminus 2^m Q} \left| \frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x_0-y|^{n-\beta}} \right|^{\frac{n}{n-\beta}} |f(y)|^{\frac{n}{n-\beta}} dy \\ & \leq \sum_{m=2}^{\infty} \frac{C2^{-m}}{|2^{m+1}Q|} \int_{2^{m+1}Q} |f(y)|^{\frac{n}{n-\beta}} dy \\ & \leq C(\mathcal{M}(f^2)(x_0))^{\frac{n}{2(n-\beta)}}. \end{aligned} \tag{8}$$

Substituting (8) into (7) arrives at

$$|I_1| \leq C(\mathcal{M}(f^2)(x_0))^{\frac{1}{2}} \leq Cb. \tag{9}$$

# Sketch of Proof of Theorem 1

Next we estimate  $I_2$ . Note that

$$\begin{aligned} |I_2| &= \left| \int_{\mathbb{R}^n \setminus 4Q} \frac{\Omega(x-y) - \Omega(x_0-y)}{|x_0-y|^{n-\beta}} \chi_\beta(|x-y|) f(y) dy \right| \\ &\leq \left( \int_{\mathbb{R}^n \setminus 4Q} \frac{|\Omega(x-y) - \Omega(x_0-y)|^{\frac{n}{n-\beta}}}{|x_0-y|^n} |f(y)|^{\frac{n}{n-\beta}} dy \right)^{\frac{n-\beta}{n}} \\ &\quad \times \left( \int_{\mathbb{R}^n \setminus 4Q} |\chi_\beta(|x-y|)|^{\frac{n}{\beta}} dy \right)^{\frac{\beta}{n}} \\ &\leq C \left( \int_{\mathbb{R}^n \setminus 4Q} \frac{|\Omega(x-y) - \Omega(x_0-y)|}{|x_0-y|^n} |f(y)|^{\frac{n}{n-\beta}} dy \right)^{\frac{n-\beta}{n}} \left[ \left( \frac{2}{\beta} \right)^n \right]^{\frac{\beta}{n}} \\ &\leq C \left( \int_{\mathbb{R}^n \setminus 4Q} \frac{|\Omega(x-y) - \Omega(x_0-y)|}{|x_0-y|^n} |f(y)|^{\frac{n}{n-\beta}} dy \right)^{\frac{n-\beta}{n}}. \end{aligned} \tag{10}$$

## Sketch of Proof of Theorem 1

$$\begin{aligned} & \int_{\mathbb{R}^n \setminus 4Q} \frac{|\Omega(x-y) - \Omega(x_0-y)|}{|x_0-y|^n} |f(y)|^{\frac{n}{n-\beta}} dy \\ &= \sum_{m=2}^{\infty} \int_{2^{m+1}Q \setminus 2^mQ} \frac{|\Omega(x-y) - \Omega(x_0-y)|}{|x_0-y|^n} |f(y)|^{\frac{n}{n-\beta}} dy. \end{aligned} \quad (11)$$

Since

$$\begin{aligned} & |\Omega(x-y) - \Omega(x_0-y)| = \left| \Omega\left(\frac{x-y}{|x-y|}\right) - \Omega\left(\frac{x_0-y}{|x_0-y|}\right) \right| \\ & \leq \omega\left(\left| \frac{x-y}{|x-y|} - \frac{x_0-y}{|x_0-y|} \right|\right) \leq \omega\left(\frac{24nl}{|y-\bar{y}|}\right) \leq \omega\left(\frac{48n}{2^m}\right) \end{aligned} \quad (12)$$

for  $y \in 2^{m+1}Q \setminus 2^mQ$ .

# Sketch of Proof of Theorem 1

Then

$$\begin{aligned} & \int_{\mathbb{R}^n \setminus 4Q} \frac{|\Omega(x-y) - \Omega(x_0-y)|}{|x_0-y|^n} |f(y)|^{\frac{n}{n-\beta}} dy \\ &= \sum_{m=2}^{\infty} \int_{2^{m+1}Q \setminus 2^mQ} \frac{|\Omega(x-y) - \Omega(x_0-y)|}{|x_0-y|^n} |f(y)|^{\frac{n}{n-\beta}} dy \\ &\leq \sum_{m=2}^{\infty} \omega\left(\frac{48n}{2^m}\right) \int_{2^{m+1}Q \setminus 2^mQ} \frac{4^n |f(y)|^{\frac{n}{n-\beta}}}{|y-\bar{y}|^n} dy \\ &\leq \sum_{m=2}^{\infty} \frac{2^{3n} \omega\left(\frac{48n}{2^m}\right)}{|2^{m+1}Q|} \int_{2^{m+1}Q} |f(y)|^{\frac{n}{n-\beta}} dy \\ &\leq C(\mathcal{M}(f^2)(x_0))^{\frac{n}{2(n-\beta)}} \sum_{m=2}^{\infty} \omega\left(\frac{48n}{2^m}\right). \end{aligned} \tag{13}$$

# Sketch of Proof of Theorem 1

For any fixed  $K > 0, 0 < \tau < 1$ , direct calculations give

$$\ln \frac{1}{\tau} \sum_{m=2}^{\infty} \omega(K\tau^m) \leq \sum_{m=2}^{\infty} \int_{K\tau^m}^{K\tau^{m-1}} \frac{\omega(t)}{t} dt = \int_0^{K\tau} \frac{\omega(t)}{t} dt.$$

Taking  $K = 48n, \tau = \frac{1}{2}$ , one has

$$\sum_{m=2}^{\infty} \omega\left(\frac{48n}{2^m}\right) \leq C \int_0^{24n} \frac{\omega(t)}{t} dt \leq C \left( \int_0^1 \frac{\omega(t)}{t} dt + \int_1^{24n} \frac{\omega(t)}{t} dt \right) \leq C. \quad (14)$$

It follows from (10), (13) and (14) that, for  $0 < \beta < \frac{n}{2}$ ,

$$|l_2| \leq C(\mathcal{M}(f^2)(x_0))^{\frac{1}{2}} \leq Cb. \quad (15)$$

# Sketch of Proof of Theorem 1

Now we estimate  $I_3$ . Direct estimates yield

$$\begin{aligned} |I_3| &= \int_{\mathbb{R}^n \setminus 4Q} \frac{\Omega(x_0 - y)}{|x_0 - y|^{n-\beta}} (\chi_\beta(|x - y|) - \chi_\beta(|x_0 - y|)) f(y) dy \\ &= \left| \sum_{m=2}^{\infty} \int_{2^{m+1}Q \setminus 2^mQ} \frac{\Omega(x_0 - y)}{|x_0 - y|^{n-\beta}} (\chi_\beta(|x - y|) - \chi_\beta(|x_0 - y|)) f(y) dy \right| \\ &\leq \sum_{m=2}^{\infty} \left( \int_{2^{m+1}Q \setminus 2^mQ} \frac{1}{|x_0 - y|^n} |f(y)|^{\frac{n}{n-\beta}} dy \right)^{\frac{n-\beta}{n}} \\ &\quad \cdot \left( \int_{2^{m+1}Q \setminus 2^mQ} |\chi_\beta(|x - y|) - \chi_\beta(|x_0 - y|)|^{\frac{n}{\beta}} dy \right)^{\frac{\beta}{n}}. \end{aligned} \tag{16}$$

# Sketch of Proof of Theorem 1

It is known that  $|\chi_\beta(|x - y|) - \chi_\beta(|x_0 - y|)| = 0$  if  $|x - y| \geq \frac{2}{\beta}$  and  $|x_0 - y| \geq \frac{2}{\beta}$ , or  $|x - y| \leq \frac{1}{\beta}$  and  $|x_0 - y| \leq \frac{1}{\beta}$ . Moreover, for  $y \in 2^{m+1}Q \setminus 2^mQ$  and  $x, x_0 \in 3Q$ , one has

$$2^m l - 3\sqrt{n}l \leq |y - \bar{y}| - |x_0 - \bar{y}| \leq |x_0 - y| \leq |y - \bar{y}| + |x_0 - \bar{y}| \leq 2^{m+1}l + 3\sqrt{n}l,$$

which implies that  $|x_0 - y| \sim 2^m l$  and similarly  $|x - y| \sim 2^m l$ . It deduces that  $\beta l \sim \frac{1}{2^m}$  in the last integral of (16) otherwise the integral will be vanished.

# Sketch of Proof of Theorem 1

Therefore, it yields

$$\begin{aligned} |I_3| &\leq C \sum_{m=2}^{\infty} \left( \frac{2^n}{(2^{m+1}l)^n} \int_{2^{m+1}Q} |f(y)|^{\frac{n}{n-\beta}} dy \right)^{\frac{n-\beta}{n}} \\ &\quad \cdot \left( \int_{2^{m+1}Q \setminus 2^m Q} |\chi'_{\beta}(t_0|x-y| + (1-t_0)|x_0-y|)|^{\frac{n}{\beta}} |x-x_0|^{\frac{n}{\beta}} dy \right)^{\frac{\beta}{n}} \\ &\leq C \sum_{m=2}^{\infty} \left( \frac{1}{(2^{m+1}l)^n} \int_{2^{m+1}Q} |f(y)|^2 dy \right)^{\frac{1}{2}} \cdot \left( \int_{2^{m+1}Q \setminus 2^m Q} (\beta l)^{\frac{n}{\beta}} dy \right)^{\frac{\beta}{n}} \\ &\leq C(\mathcal{M}(f^2)(x_0))^{\frac{1}{2}} \sum_{m=2}^{\infty} \left( \int_{2^{m+1}Q \setminus 2^m Q} (\beta l)^{\frac{n}{\beta}} dy \right)^{\frac{\beta}{n}} \\ &\leq C(\mathcal{M}(f^2)(x_0))^{\frac{1}{2}} \sum_{m=2}^{\infty} \frac{1}{2^m} (2^m l)^{\beta} \\ &\leq C(\mathcal{M}(f^2)(x_0))^{\frac{1}{2}} \sum_{m=2}^{\infty} \frac{1}{2^m} \left( \frac{1}{\beta} \right)^{\beta} \leq C(\mathcal{M}(f^2)(x_0))^{\frac{1}{2}} \leq Cb, \end{aligned}$$



# Sketch of Proof of Theorem 1

Combining (6) with (9), (15) and (17), we finish the proof of (3) and hence **the claim** holds true. The proof of the Lemma is finished.

# Sketch of Proof of Theorem 1

Based on Lemma 1, we have

## Lemma 2(Jiu-Li-Yu, 2020)

Assume that  $f \in L^2(\mathbb{R}^n)$ . Then for any  $\mu > 0$ , we can choose a  $0 < \delta \leq 1$  depending on  $\mu$  such that

$$\begin{aligned} & |\{x \in \mathbb{R}^n : \mathcal{M}(|T_1 f|^2)(x) > \lambda N^2\}| \\ & \leq 2\mu(|\{x \in \mathbb{R}^n : \mathcal{M}(|T_1 f|^2)(x) > \lambda\}| + |\{x \in \mathbb{R}^n : \mathcal{M}(f^2)(x) > \lambda \delta^2\}|). \end{aligned}$$

holds for  $\lambda > 0$  and  $0 < \beta < 1$ , where  $N > 1$  is some fixed constant (In fact,  $\frac{N}{4} = \max\{4 \cdot 5^n, (2 + C)^2\}$ ).

The Lemma implies that the distribution function

$|\{x \in \mathbb{R}^n : \mathcal{M}(|T_1 f|^2)(x) > \lambda N^k\}|$  decays fast as  $k$  becomes large, for  $\lambda > 0$  and  $N > 1$  in some sense.

# Sketch of Proof of Theorem 1

Using Lemma 2 above, we prove our main result as follows.

$$\begin{aligned} \int_{\mathbb{R}^n} (\mathcal{M}(|T_1 f|^2)(x))^{\frac{q}{2}} dx &= \frac{q}{2} \int_0^\infty t^{\frac{q}{2}-1} |\{x \in \mathbb{R}^n : \mathcal{M}(|T_1 f|^2)(x) > t\}| dt \\ &= \frac{q}{2} N^q \int_0^\infty t^{\frac{q}{2}-1} |\{x \in \mathbb{R}^n : \mathcal{M}(|T_1 f|^2)(x) > N^2 t\}| dt \\ &\leq \mu q N^q \int_0^\infty t^{\frac{q}{2}-1} (|\{x \in \mathbb{R}^n : \mathcal{M}(|T_1 f|^2)(x) > t\}| \\ &\quad + |\{x \in \mathbb{R}^n : \mathcal{M}(f^2)(x) > t\delta^2\}|) dt. \end{aligned}$$

# Sketch of Proof of Theorem 1

Let  $\mu = \frac{1}{2qN^q}$ . It follows that

$$\begin{aligned}\int_{\mathbb{R}^n} (\mathcal{M}(|T_1 f|^2)(x))^{\frac{q}{2}} dx &\leq \int_0^\infty t^{\frac{q}{2}-1} |\{x \in \mathbb{R}^n : \mathcal{M}(f^2)(x) > t\delta^2\}| dt \\ &\leq \frac{1}{\delta^q} \int_0^\infty t^{\frac{q}{2}-1} |\{x \in \mathbb{R}^n : \mathcal{M}(f^2)(x) > t\}| dt \\ &\leq \frac{1}{\delta^q} (C_{\frac{q}{2}})^{\frac{q}{2}} \int_{\mathbb{R}^n} |f|^q dx,\end{aligned}$$

where in the last inequality we used the strong  $(\frac{q}{2}, \frac{q}{2})$  type inequality of the maximal function for  $2 < q < \infty$ . Since  $|T_1 f|^2(x) \leq \mathcal{M}(|T_1 f|^2)(x)$  a.e., it yields that  $T_1$  is of strong type  $(q, q)$  for  $2 < q < \infty$ .

Using the duality method, we obtain that  $T_1$  is of strong type  $(q, q)$  for  $1 < q < 2$ . The proof of Theorem 1 is complete.

### 3. An application in SQG

The Generalized (Modified) SQG:

$$\begin{cases} \omega_t + u \cdot \nabla \omega = 0, & (x, t) \in \mathbb{R}^2 \times \mathbb{R}^+, \\ u = \nabla^\perp (-\Delta)^{-1+\alpha} \omega, \\ \omega(x, 0) = \omega_0, \end{cases} \quad (\text{SQG})$$

Here  $0 \leq \alpha \leq \frac{1}{2}$ ,  $\nabla^\perp = (-\partial_2, \partial_1)$  and  $((-\Delta)^{-1+\alpha} f)^\wedge = |\xi|^{-2+2\alpha} \hat{f}$ .

- $\alpha = 0$ , 2D incompressible Euler equations.
- $\alpha = \frac{1}{2}$ , SQG (Surface Quasi-geostrophic) equation. Physically, it is a kind of approximation for nonhomogeneous fluid flow in a rapidly rotating 3D half-space. Mathematically, the formation of singular solution for the SQG would be similar to the 3D Euler equations.
- $0 < \alpha < \frac{1}{2}$ : Generalized (Modified) SQG.

## Some related results

- P. Constantin, A. Majda, E. Tabak: Singular front formation in a model for quasigeostrophic flow *Nonlinearity*, 1994.

### Theorem 3(Jiu-Yu-Zhang (2020))

Let  $\alpha = \frac{1}{2}$ . For every  $\omega_0 \in H^s(\mathbb{R}^2)$ ,  $s > 2$ , there exists a small time  $T = T(\|\omega_0\|_{H^s(\mathbb{R}^2)})$  such that (SQG) admits a unique solution  $\omega \in C([0, T]; H^s(\mathbb{R}^2))$ .

### Theorem 4(Jiu-Yu-Zhang (2020))

Let  $0 < \alpha < \frac{1}{2}$ . For every  $\omega_0 \in H^s(\mathbb{R}^2)$ ,  $s > 1 + 2\alpha$ , there exists a small time  $T = T(\|\omega_0\|_{H^s(\mathbb{R}^2)})$  such that (SQG) admits a unique solution  $\omega \in C([0, T]; H^s(\mathbb{R}^2))$ .

- H.Inci, On the well-posedness of the inviscid SQG equation, JDE, 2020:  $\alpha = \frac{1}{2}$ , different approach.

## Some related results

- G. Luo, T. Y. Hou, Potentially singular solutions of the 3D axisymmetric Euler equations, [Proc.Nat.Acad.Sci.USA, 2014](#).
- A. Kiselev, V. Sverak, Small scale creation for solutions of the incompressible Euler equations, [Ann. Math. 2014](#)
- A. Kiselev, Y. Yao, A. Zlatos, Local Regularity for the modified SQG patch equation, [CPAM 2017](#).
- A. Kiselev, L. Ryzhik, Y. Yao, A. Zlatos, Finite time singularity formation for the modified SQG patch equation, [Ann. Math., 2016](#):  
 $0 < \alpha < \frac{1}{24}$ .

### Theorem 5 (H. Yu, X. Zheng, Q. Jiu, ARMA 2019)

Let  $0 < \alpha < \alpha_0 = 1/2$ . Let  $\omega^{\alpha_0}$  be a solution of (SQG) for  $0 \leq t \leq T$  with  $u^{\alpha_0} = \nabla^\perp (-\Delta)^{-1+\alpha_0} \omega^{\alpha_0}$  and  $\omega_0 \in H^{s+1} \cap L^1$ ,  $s > 2$ . Then, there exists  $\delta > 0$  such that if  $0 < \alpha_0 - \alpha < \delta$ , the solution  $\omega^\alpha$  to (SQG) with  $u^\alpha = \nabla^\perp (-\Delta)^{-1+\alpha} \omega^\alpha$  and the same initial data is smooth on  $[0, T]$ . Moreover, it holds that

$$\|\omega^\alpha(t) - \omega^{\alpha_0}(t)\|_{H^s} \leq C((1/2 - \alpha) + (1/2 - \alpha)|\ln(1/2 - \alpha)|^2).$$



The proof is associated with an uniform estimate of the singular integral:

$$u(x) = \nabla^\perp (-\Delta)^{-1+\alpha} \omega = \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \omega(y) dy,$$

where  $x^\perp = (-x_2, x_1)$ ,  $0 \leq \alpha \leq \frac{1}{2}$ .

If we take  $n = 2, \beta = 1 - 2\alpha$ , this can be viewed as a special case of the following singular integral

$$T_j f(x) = \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^{n+1-\beta}} f(y) dy,$$

where  $0 < \beta < n, j = 1, 2, \dots, n$ , which is exactly (\*) with  $\Omega(x) = \frac{x_j}{|x|}$ .

# Proof of Theorem 3

Denote

$$\bar{\omega} = \omega^\alpha - \omega^{\alpha_0} \quad \text{and} \quad \bar{u} = u^\alpha - u^{\alpha_0}.$$

$\bar{u}$  is divided into

$$\begin{aligned} \bar{u} &= u^\alpha - u^{\alpha_0} \\ &= \nabla^\perp (-\Delta)^{-1+\alpha} \bar{\omega} + (\nabla^\perp (-\Delta)^{-1+\alpha} - \nabla^\perp (-\Delta)^{-1+\alpha_0}) \omega^{\alpha_0} \\ &:= \bar{u}_I + \bar{u}_{II}. \end{aligned}$$

# Proof of Theorem 5

Lemma 3(H. Yu, X. Zheng, Q. Jiu, ARMA 2019)

There exists a constant  $C = C(n, s)$  independent of  $\beta$  such that

$$\|T_1 f\|_{H^s} \leq C \|f\|_{H^s}, \quad s \geq 0, 0 < \beta \leq 1 - \beta_0;$$

$$\|T_2 f\|_{L^2} \leq C \frac{\beta^{\frac{n}{2}}}{\sqrt{n-2\beta}} \|f\|_{L^1}, \quad 0 < \beta < \frac{n}{2};$$

$$\|T_2 f\|_{\dot{H}^s} \leq C \frac{\beta}{1-\beta} \|f\|_{\dot{H}^{s-1}}, \quad s \geq 1, 0 < \beta < 1.$$

## Proof of Theorem 5

$$\bar{\omega}_t + (u^{\alpha_0} \cdot \nabla) \bar{\omega} + (\bar{u} \cdot \nabla) \bar{\omega} + (\bar{u} \cdot \nabla) \omega^{\alpha_0} = 0.$$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\bar{\omega}(t)\|_{H^s}^2 &= - \int_{\mathbb{R}^2} J^s(u^{\alpha_0} \cdot \nabla \bar{\omega}) J^s \bar{\omega} \, dx - \int_{\mathbb{R}^2} J^s(\bar{u} \cdot \nabla \bar{\omega}) J^s \bar{\omega} \, dx \\ &\quad - \int_{\mathbb{R}^2} J^s(\bar{u} \cdot \nabla \omega^{\alpha_0}) J^s \bar{\omega} \, dx. \end{aligned}$$

Here  $J^s = (I - \Delta)^{\frac{s}{2}}$ ,  $\widehat{J^s f}(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi)$ ,  $s \in \mathbb{R}$ . Then

$$\begin{aligned} \frac{d}{dt} \|\bar{\omega}\|_{H^s} &\leq \|\bar{\omega}\|_{H^s} \|u^{\alpha_0}\|_{H^s} + \|\bar{\omega}\|_{H^s} (\|\bar{u}_I\|_{H^s} + \|\bar{u}_{II}\|_{H^s}) \\ &\quad + \|\omega^{\alpha_0}\|_{H^{s+1}} (\|\bar{u}_I\|_{H^s} + \|\bar{u}_{II}\|_{H^s}). \end{aligned}$$

# Proof of Theorem 5

Taking  $\beta = 1 - 2\alpha$ , by Lemma 3, we obtain

$$\|\bar{u}_I\|_{H^s} \leq C(\|\bar{\omega}\|_{H^s} + (1 - 2\alpha)\|\bar{\omega}\|_{L^1}).$$

Using Lemma 3 above again ( $\beta = 1 - 2\alpha$  and  $\beta = 0$  respectively), we have

$$\|\bar{u}_{II}\|_{H^s} \leq C(\|\omega^{\alpha_0}\|_{H^s} + \|\omega^{\alpha_0}\|_{L^1}).$$

It follows that

$$\begin{aligned} & \|\bar{\omega}\|_{H^s}(\|\bar{u}_I\|_{H^s} + \|\bar{u}_{II}\|_{H^s}) \\ & \leq C\|\bar{\omega}\|_{H^s}^2 + C\|\bar{\omega}\|_{H^s}(\|\omega^{\alpha_0}\|_{H^s} + \|\omega^{\alpha_0}\|_{L^1} + \|\omega^\alpha\|_{L^1}). \end{aligned}$$

# Proof of Theorem 5

To estimate the term  $\|\omega^{\alpha_0}\|_{H^{s+1}}(\|\bar{u}_I\|_{H^s} + \|\bar{u}_{II}\|_{H^s})$ , we use the following decomposition:

$$\begin{aligned} & J^s \bar{u}_{II} \\ &= \int \left( \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} - \frac{(x-y)^\perp}{|x-y|^{2+2\alpha_0}} \right) J^s \omega^{\alpha_0}(y) dy \\ &= \left( \int_{|x-y| \leq \epsilon} + \int_{1 > |x-y| \geq \epsilon} + \int_{|x-y| \geq 1} \right) \left( \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} - \frac{(x-y)^\perp}{|x-y|^{2+2\alpha_0}} \right) J^s \omega^{\alpha_0} dy \\ &= H_1 + H_2 + H_3. \end{aligned}$$

where  $0 < \epsilon < 1$  is to be determined.

## Proof of Theorem 5

Using the fact that  $\int_{|x|=1} \frac{x^\perp}{|x|^{2+2\alpha}} ds = \int_{|x|=1} \frac{x^\perp}{|x|^3} ds = 0$ ,

$$H_1 = \int_{|z| \leq \epsilon} \left( \frac{z^\perp}{|z|^{2+2\alpha}} - \frac{z^\perp}{|z|^3} \right) (J^s \omega^{\alpha_0}(x-z) - J^s \omega^{\alpha_0}(x)) dz.$$

From the mean value theorem, we deduce that

$$\begin{aligned} \|H_1\|_{L^2} &\leq \int_0^1 \int_{|z| \leq \epsilon} \left( \frac{1}{|z|^{2\alpha}} + \frac{1}{|z|} \right) \|\nabla J^s \omega^{\alpha_0}(\cdot - \tau z)\|_{L^2} dz d\tau \\ &\leq C \left( \frac{1}{2-2\alpha} \epsilon^{2-2\alpha} + \epsilon \right) \|J^{s+1} \omega^{\alpha_0}\|_{L^2}. \end{aligned}$$

For  $2+2\alpha \leq \xi \leq 3$ , we estimate  $H_2$  as follows,

$$\begin{aligned} \|H_2\|_{L^2} &= \left( \frac{1}{2} - \alpha \right) \left\| \int_{1 > |x-y| \geq \epsilon} \frac{(x-y)^\perp (|x-y|^\xi \ln |x-y|)}{|x-y|^{2+2\alpha} |x-y|^{2+2\alpha_0}} J^s \omega^{\alpha_0}(y) dy \right\|_{L^2} \\ &\leq \left( \frac{1}{2} - \alpha \right) \left\| \int_{1 > |x-y| \geq \epsilon} \frac{|\ln |x-y||}{|x-y|^2} |J^s \omega^{\alpha_0}(y)| dy \right\|_{L^2} \\ &\leq \left( \frac{1}{2} - \alpha \right) |\ln \epsilon|^2 \|J^s \omega^{\alpha_0}\|_{L^2}. \end{aligned}$$

# Proof of Theorem 5

By choosing some suitable  $\frac{2(s+1)}{s+2} \leq p < \frac{2}{2-2\alpha+\sigma} < 2$ ,

$$\begin{aligned}\|H_3\|_{L^2} &\leq \left(\frac{1}{2} - \alpha\right) \left\| \int_{|x-y|\geq 1} \frac{|\ln|x-y||}{|x-y|^{1+2\alpha}} |J^s \omega^{\alpha_0}(y)| dy \right\|_{L^2} \\ &\leq \left(\frac{1}{2} - \alpha\right) \|J^s \omega^{\alpha_0}\|_{L^p} \left( \int_{|x-y|\geq 1} \left( \frac{|\ln|x-y||}{|x-y|^{1+2\alpha}} \right)^q dy \right)^{\frac{1}{q}} \\ &\leq \left(\frac{1}{2} - \alpha\right) \|\omega^{\alpha_0}\|_{H^{s+1} \cap L^1}.\end{aligned}$$

By choosing  $\epsilon = \left(\frac{1}{2} - \alpha\right)$ , we get

$$\|\bar{u}_{II}\|_{H^s} \leq CL(\alpha) (\|\omega^{\alpha_0}\|_{H^{s+1}}^2 + \|\omega^{\alpha_0}\|_{L^1}^2).$$

Here  $L(\alpha) := \left(\frac{1}{2} - \alpha\right)^{2-2\alpha} + \left(\frac{1}{2} - \alpha\right) \left| \log \left(\frac{1}{2} - \alpha\right) \right|^2 + \left(\frac{1}{2} - \alpha\right).$



# Proof of Theorem 5

Combining above all, we have

$$\begin{aligned} \frac{d}{dt} \|\bar{\omega}(t)\|_{H^s} &\leq \|\bar{\omega}\|_{H^s} (\|\omega^{\alpha_0}\|_{H^{s+1}} + \|\omega^{\alpha_0}\|_{L^1} + \|\omega^\alpha\|_{L^1}) + \|\bar{\omega}\|_{H^s}^2 \\ &\quad + CL(\alpha) (\|\omega^{\alpha_0}\|_{H^{s+1}}^2 + \|\omega^{\alpha_0}\|_{L^1}^2 + \|\omega^\alpha\|_{L^1}^2). \end{aligned}$$

Then we obtain

$$\|\bar{\omega}\|_{H^s} \leq C \left( \left( \frac{1}{2} - \alpha \right) + \left( \frac{1}{2} - \alpha \right) \left| \ln \left( \frac{1}{2} - \alpha \right) \right|^2 \right).$$

Thanks for your attentions!