19.08.2022. (36)

It remains to show when k : a field, $\mathcal{O} = k[\varepsilon]$.

$$\boxed{\dim_k H^1(A, m\{\tau\}) = d}$$

in the derived category

$$R\mathrm{Hom}_{A\otimes A}(A, m\{\tau\}) \simeq T_{A/\mathbb{F}_p} \overset{L}{\underset{A\otimes A}{\otimes}} m\{\tau\}[-1]$$

$$\wr| \quad H^0, H^n = 0 \quad n \geq 2$$

$$\mathrm{Ext}^1(A, m\{\tau\})[-1]$$

where $T_{A/\mathbb{F}_p} = \mathrm{Hom}_A(\Omega^1_{A/\mathbb{F}_p}, A)$: tangent module of A. : loc. free rank 1 / A (right A-module)

viewed as right $A\otimes A$-module via $A \otimes A \to A$

$\mathcal{O} = k[\varepsilon]$ $\quad m = k\cdot\underline{\underline{\varepsilon}}$. $\quad m\{\tau\} \cong \underline{\underline{k\{\tau\}}}$ as module.

$$\mathrm{Def}_\varphi(k[\varepsilon]) = H^1(A, k\{\tau\}) \simeq T_{A/\mathbb{F}_p} \overset{(L)}{\underset{A\otimes A}{\otimes}} k\{\tau\},$$

$k\{\tau\}$: (A, A)-module structure given by

$$a \cdot P(\tau) b = \theta(a) \cdot P(\tau) \cdot \varphi(b)$$

$\Rightarrow$ $A\otimes A$-module structure of $k\{\tau\}$ coming from $k \underset{\mathbb{F}_k}{\otimes} A$

via $$A \otimes A \xrightarrow{\theta \otimes \mathrm{id}_A} \underline{\underline{k \otimes A \circlearrowright k\{\tau\}}}$$

Lemma 5 $\varphi : A \to k\{\tau\}$ Drinfeld A-module of rank d / k

Let $k\{\tau\}$ ($= m\{\tau\}$) be the $k \otimes_{\mathbb{F}_p} A$-module def'd by $(\alpha \otimes a)(P(\tau)) = \alpha \cdot P(\tau) \cdot \varphi(a)$. Then $k\{\tau\}$ is a locally free $k \otimes_{\mathbb{F}_p} A$-mod. of rank d.

—

$$\left(N \otimes_A M \, , \quad M : A/I\text{-module} \Rightarrow N/IN \otimes_{A/I} M \right)$$

$$\boxed{T_{A/\mathbb{F}_p} \overset{L}{\otimes}_{\underline{\underline{A}}} \left(\underline{\frac{A \otimes A}{J}} \right) \overset{L}{\otimes}_{A \otimes A} k\{\tau\} = T \otimes_{\underline{A},\theta} k \otimes_{\theta_k, A_k} k\{\tau\} \simeq k^d}$$

$$A \otimes A \longrightarrow A \, . \qquad J = (a \otimes 1 - 1 \otimes a : a \in A)$$

$$A \otimes A/_J \simeq A \, , \qquad A \simeq A \otimes A/_J$$

$$a \mapsto [a \otimes 1]$$

$$\frac{A \otimes A}{J} \otimes_{A \otimes A} k\{\tau\} = \frac{k \otimes A}{J'} \otimes_{k \otimes A} k\{\tau\}$$

$$k \otimes A \xrightarrow{\mathrm{id} \otimes \theta} k \, , \qquad J' = \ker \theta' = (\theta(a) \otimes 1 - 1 \otimes a \, ; \, a \in A) \Rightarrow \frac{k \otimes A}{J'} \simeq k$$

θ_k

$\theta_k : A_k = k \otimes_{\mathbb{F}_p} A \to k$, $\alpha \mapsto \alpha \quad \alpha \in k$

$a \mapsto \theta(a) \quad a \in A.$

base change $\theta : A \to k$ over k

$\Rightarrow$ $M_I^d \to \operatorname{Spec} \mathbb{F}_p$: smooth of dim d.

$\textcircled{H}$: $M_I^d \longrightarrow \operatorname{Spec} A[I^{-1}]$, $\varphi : A \to k\{\tau\}$: k-point of M_I^d.

The tangent spaces at φ

$$T_{A/\mathbb{F}_p} \underset{A\otimes A}{\otimes} k\{\tau\} \xrightarrow{\mathrm{id}\otimes\partial} T_{A/\mathbb{F}_p} \underset{A\otimes A}{\otimes} k = T_{A/\mathbb{F}_p} \underset{A,\theta}{\otimes} k$$

$$k\{\tau\} \xrightarrow{\partial} k$$

$\Rightarrow$ $\textcircled{H}$: smooth morphism of rel. dim $d-1$. $\square$.

Proof of Lemma 5.

Choose $a \neq 0 \in A$. $\mathbb{F}_p[a]$: polynomial ring.

$A \supset \mathbb{F}_p[a]$: A : torsion-free, finite $\mathbb{F}_p[a]$-module

: free $\mathbb{F}_p[a]$-module of rank $\deg_{\mathbb{F}_p} a \; (:= \dim_{\mathbb{F}_p} A/(a))$

(if $A \simeq \mathbb{F}_p[a]^r$. $A/(a) \simeq \mathbb{F}_p^r \Rightarrow r = \dim_{\mathbb{F}_p} A/(a)$).

$\varphi(a) \in k\{\tau\}$. $\forall\, P(\tau) \in k\{\tau\}$ using Euclidean algorithm

$$P(\tau) = P_1(\tau)\,\varphi(a) + R_1(\tau)$$

$$P_1(\tau) = P_2(\tau)\,\varphi(a) + R_2(\tau)$$

$$\vdots$$

$\mathbb{F}_p[a] = \mathbb{F}_p\langle 1, a, a^2, \cdots \rangle$

$\varphi^m(a) = \varphi(a^m)$

where $\deg R_m(\tau) < \deg \varphi(a) = d \cdot \deg_{\mathbb{F}_p} a$

So $$P(\tau) = \sum_{m \geq 0} \sum_{n=0}^{d\cdot \deg_{\mathbb{F}_p}(a) - 1} a_{m,n}\, \tau^n \cdot \varphi^m(a), \qquad \{1, \tau, \cdots, \tau^{d\cdot\deg_{\mathbb{F}_p}(a)-1}\}$$

So $k\{\tau\}$: a free $k \otimes \mathbb{F}_p[a]$-module rank $1, \tau, \cdots \tau^{d\cdot\deg_{\mathbb{F}_p}(a)-1}$

$k\{\tau\}$ as $k \otimes A$ - module

If $k \otimes A$: integral domain $\Rightarrow$ Dedekind domain.

$k\{\tau\}$: torsion - free $k \otimes A$ - module.

then $k\{\tau\}$: loc. free of rank $\dfrac{d \cdot \deg_{\mathbb{F}_p}(a)}{\deg_{\mathbb{F}_p}(a)} = d$.

Otherwise : let $\mathbb{F}_{p^r} \subset A$ the constant subfield $\subset k$

$$k \underset{\mathbb{F}_p}{\otimes} \mathbb{F}_{p^r} \simeq \prod_{i=0}^{r-1} k \qquad \alpha \otimes a \longmapsto \left(\alpha \cdot \theta(a)^{p^i}\right)$$

$$k \underset{\mathbb{F}_p}{\otimes} A = \left(k \otimes \mathbb{F}_{p^r}\right) \underset{\mathbb{F}_{p^r}}{\otimes} A \simeq \prod_{i=0}^{r-1} \left(k \underset{\theta^{p^i}, \mathbb{F}_{p^r}}{\otimes} A\right)$$

$$k\{\tau\} \simeq \bigoplus_{i=0}^{r-1} k\{\tau^r\} \cdot \tau^i$$

Each $k\{\tau^r\}\tau^i$: locally free of rank $d \,/\, k \underset{\theta^{p^i}, \mathbb{F}_{p^r}}{\otimes} A$

$\Rightarrow$ $k\{\tau\}$: $\quad$ " $\quad$ of rank $d \,/\, k \underset{\mathbb{F}_p}{\otimes} A$

(IV) Endomorphism algebras of Drinfeld modules.

$\mathrm{End}(\mathbb{G}_{a,k})$:

k : field $\supset \mathbb{F}_p$. $\mathbb{G}_{a/k}$ $\mathrm{End}(\mathbb{G}_a) = k\{\tau\}$ $\tau(t) = t^p$ $(\tau = \tau_p)$

$$\begin{tikzcd} \mathbb{G}_a \arrow[r, "u"] \arrow[rr, bend right, "v"'] & \mathbb{G}_a \arrow[r, dashed, "w"] & \mathbb{G}_a \end{tikzcd}, \qquad G = \ker u$$

$v(G) = 0 \;(\Leftrightarrow G \subseteq \ker v) \Longleftrightarrow v = w \circ u$ (i.e v is right divisible by u)

$$0 \to G^{\circ} \to G \to G^{et} \to 0$$

Thm (Cartier) Every group scheme of finite type over a field of char 0 is reduced.

$$u = \underbrace{(a_0 + a_1\tau + \cdots + a_n\tau^n)}_{u^{et}} \cdot \tau^h. \qquad a_0 \neq 0.$$

$G^{\circ} = \ker \tau^h = \mathrm{Spec}\, k[t]/(t^{p^h})$.

$G^{et} = \ker k[T]/\underbrace{(a_0T + a_1T^p + \cdots + a_nT^{p^n})}_{u^{et}(T)}$, $G = \mathrm{Spec}\, k[t]/u(t)$

$$\frac{k[t]}{u(t)} \leftarrow \frac{k[T]}{u^{et}(T)} \qquad t^{p^h} \leftarrow T$$

Lemma 1 $G \subset \mathbb{G}_{a/k}$ finite k-subgroup scheme. Then $\exists\, u \in k\{\tau\}$ s.t. $G = \ker u$. Moreover, if $v \in k\{\tau\}$, $G \subseteq \ker v$, then $\exists\, w \in k\{\tau\}$ s.t. $v = w \circ u$.

Pf: $G = \operatorname{Spec} k[t]/f(t)$. $f(t)$: monic poly. of deg n.

$t_1 = t \otimes 1$
$t_2 = 1 \otimes t$

$$\Delta: k[t]/f(t) \longrightarrow k[t]/f(t) \otimes k[t]/f(t) = k[t_1,t_2]/(f(t_1), f(t_2))$$

$$\Delta(f(t)) = f(t_1 + t_2) \in (f(t_1), f(t_2))$$

$$f(t_1+t_2) = f(t_1)\, P(t_1,t_2) + f(t_2)\, Q(t_1,t_2)$$

We may assume that P, Q are minimal tot. degree, say of deg r.

Want to show $r = 0$. Suppose $r > 0$.

$P = \sum^r A_i$, $Q = \sum^r B_i$, A_i, B_i : homog poly of deg i.

top: $t_1^n \cdot A_r + t_2^n \cdot B_r \Rightarrow t_2^n \mid A_r$, $t_1^n \mid B_r$

$A_r = t_2^n \cdot A'$, $B_r = t_1^n \cdot (-A')$

Put $P' = P - f(t_2) A'$, $Q' = Q + f(t_1) A'$

$f(t_1) \cdot P + f(t_2)\, Q = f(t_1) \cdot P' + f(t_2)\, Q'$.

$\deg Q' < \deg Q$
$\deg P' < \deg P$

$\Rightarrow P = a$, $Q = b$, $a, b \in k$. $f(t_1 + t_2) = f(t_1) a + f(t_2) b$

deg n: $(t_1 + t_2)^n = a\, t_1^n + b\, t_2^n \Rightarrow a = b = 1$.

f: additive. $\exists\, u \in k\{\tau\}$, $u(t) = f(t)$. $\square$

Lemma 2. φ, φ' : Drinfeld A-modules $/ k$. $u : \varphi \to \varphi'$: isogeny

Then $\exists\, a \neq 0 \in A$ and an isogeny $u' : \varphi' \to \varphi$ s.t. $u' \circ u = \varphi_a$.

Pf: $G = \ker u$, Claim $\exists\, a \neq 0 \in A$ s.t. $\varphi_a(G) = 0$

$G^{et}(\bar{k})$: finite A-module. $\exists\, a_1 \in A$ s.t. $\varphi_{a_1}(G^{et}(\bar{k})) = 0$

$\varphi_{a_1}(G^{et}) = 0$

If $G^{\circ} \neq \{0\}$, then k has finite $\mathrm{char}_A k = v$.

Choose a_2, $v(a_2)$: big enough. $\Rightarrow \varphi_{a_2}(G^{\circ}) = 0$

Let $a = a_1 a_2$, $\varphi_a(G) = 0$.

By Lemma 1. $\exists\, u' \in \mathrm{End}(\mathbb{G}_a)$. $u' \circ u = \varphi_a$

$b \in A$. $\varphi_b \circ (\underbrace{u' \circ u}_{\varphi_a}) = u' \circ \underline{\underline{u}} \circ \varphi_b = u' \circ \varphi_b \circ u$

$\Rightarrow \varphi_b \circ u' = u' \circ \varphi_b$ u' : isogeny.

φ : Drinfeld A-module of rank d / k. $\varphi : A \hookrightarrow \mathrm{End}(\mathbb{G}_a)$, $\mathrm{End}(\mathbb{G}_a) \supseteq \mathrm{End}(\varphi)$, $A \hookrightarrow \mathrm{End}(\varphi)$

$\mathrm{End}(\varphi) = \mathrm{End}_A(\mathbb{G}_a)$, $\mathrm{End}^{\circ}(\varphi) := F \otimes_A \mathrm{End}(\varphi)$

$A \subseteq Z(\mathrm{End}(\varphi))$, endomorphism algebra of φ. $/ k$.

$$\varphi_a = \sum_{i=0}^{d \cdot \deg(a)} \varphi_{a,i} \tau^i \qquad (\tau = \tau_q)$$

$\deg(\varphi_a) = d \cdot \deg a$.

$\deg: \mathrm{End}^\circ(\varphi) \longrightarrow \mathbb{Z} \cup \{-\infty\}$ $\quad \deg(\frac{a}{b}) := \deg(a) - \deg(b)$

$\deg(0) = -\infty$

$\|\cdot\|: \mathrm{End}^\circ(\varphi) \longrightarrow \mathbb{R}$.

$\|u\| = q^{\deg(u)/d}$. $\quad \deg(f+g) \leq \max\{\deg f, \deg g\}$.

Note $a \in A$ $\quad \|a\| = q^{\deg(\varphi_a)/d} = q^{\deg(a)} = q^{-\deg(\infty)\,\infty(a)} = q_\infty^{-\infty(a)}$

$= |a|_\infty$ $\quad$ Completion $\to F_\infty$ $\quad q_\infty = \# k(\infty)$

Then: $\forall\, u, v \in \mathrm{End}^\circ(\varphi)$, $a \in F$.

(i) $\|u\| \geq 0$ & $\|u\| = 0 \iff u = 0$

(ii) $\|a \cdot u\| = |a|_\infty \|u\|$.

(iii) $\|u+v\| \leq \max\{\|u\|, \|v\|\}$.

(iv) $\|u \cdot v\| = \|u\| \cdot \|v\|$.

$\|\cdot\|$: norm.

Prop 3: $\mathrm{End}(\varphi)$: fin. gen. projective A-module of rank $\leq d^2$.

and $\mathrm{End}^\circ(\varphi) \otimes_F F_\infty$: division algebra$/F_\infty$ (definite at ∞)

Pf: (i) & (ii) $\Rightarrow$ $\mathrm{End}(\varphi)$: torsion-free.

Let $V \subset \mathrm{End}^\circ(\varphi)$: a finite-dim'l F-subspace.

$$(V, \|\ \|) \xrightarrow{\text{Completion}} (\underset{\substack{\| \wr \\ V_\infty}}{F_\infty \otimes_F V}, \|\ \|) \qquad \|\cdot\|: \text{a norm}$$

F_∞-v.s.

Note $\mathrm{End}(\varphi)$ has bounded norm.

$\Lambda = V \cap \mathrm{End}(\varphi)$. $\Lambda = V_\infty \cap \mathrm{End}(\varphi) \subset V_\infty$: discrete.

$\Rightarrow$ Λ: fin. gen, torsion-forsion. $\Rightarrow$ projective of finite rank.

Let $a \neq 0 \in A$: prime to $\mathrm{char}_A k$.

$$f \in \Lambda, \quad f: \underset{\substack{\| \\ (A/a)^d}}{\varphi[a](\bar{k})} \longrightarrow \varphi[a](\bar{k})$$

$$A/a \otimes \Lambda \longrightarrow \mathrm{End}_{A/a}(\varphi[a](\bar{k})) \simeq \mathrm{Mat}_d(A/a)$$

Injective: $f(\underset{\substack{\| \\ \ker(\varphi_a)}}{\varphi[a](\bar{k})}) = 0 \Rightarrow f = \varphi_a \circ f'$

$f = 0$ in $A/a \otimes \Lambda$.

$A/a \otimes \Lambda$ free A/a-module of rank $\leq d^2$

Λ: of rank $\leq d^2$

$\mathrm{End}^\circ(\varphi)$: fin dim'l of dim $\leq d^2$.

$\mathrm{End}(\varphi)$ :: f. g. prj A-module of rank $\leq d^2$.

$$(\mathrm{End}^\circ(\varphi), \|\ \|) \xrightarrow{\text{completion}} (F_\infty \otimes \mathrm{End}^\circ(\varphi), \|\cdot\|)$$

with (i)–(iv) $u \neq 0$, $v \neq 0$

$\Rightarrow u \cdot v \neq 0$ □

$F_\infty \otimes \mathrm{End}^\circ(\varphi)$: division alg / F_∞

(not product of simple factors)

$\Rightarrow \|\cdot\|$ = the unique extn of $|\ |_\infty$ on F.