

國家天元數學中部中心  
算術幾何短期課程

Introduction to geometry of  
Siegel modular varieties mod  $p$ .

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## References:

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## § Introduction.

Central topics: abelian varieties and Siegel modular varieties

Interested in: geometry of Siegel modular varieties via

regarding them as moduli spaces of abelian varieties, particularly in char  $p > 0$ .

Goal: introduce Newton stratification and EO stratification. (2)

description of these classifications.

geometry of these strata, - examples, and ideas of some proofs.

introduce basic tools of Dieudonné modules, and of reductive groups.

supersingular locus, supersingular EO strata, connection with Deligne-Lusztig varieties  $\leftarrow$  more input from group theory,

introduce KR stratification and some of its relation with EO stratification.

## § Preliminaries on abelian varieties.

1.1 Def. An abelian variety over a field  $k$  is a connected proper smooth algebraic group over  $k$ .

Thm: Every abelian variety  $A/k$  is a commutative alg gp and is a projective alg var/ $k$ .

Over  $\mathbb{C}$ ,  $A(\mathbb{C}) = \mathbb{C}^g / \Lambda$  complex torus. and  $A$  is determined by its associated complex torus. More precisely, we have

$$\mathrm{Hom}(A, B) \xrightarrow{\sim} \mathrm{Hom}(A(\mathbb{C}), B(\mathbb{C})) \quad (\text{GAGA})$$

$$\text{So } A \simeq B \iff A(\mathbb{C}) \simeq B(\mathbb{C})$$

We have a complete understanding of all line bundles on  $\textcircled{3}$   
a complex torus  $V/\Lambda$ . (Appel-Humbert's Thm):

every line bundle can be constructed by a pair  $(H, \alpha)$ ,

where  $H$ : Hermitian form on  $V$ . s.t  $E = \text{Im} H: \Lambda \times \Lambda \rightarrow \mathbb{Z}$ .

$$\alpha: \Lambda \rightarrow \mathbb{C}^\times = \{z: |z|=1\} \text{ s.t. } \alpha(u_1 + u_2) = \alpha(u_1) \alpha(u_2) \cdot e^{\pi i E(u_1, u_2)}$$

By Lefschetz's Thm: complex abelian varieties are  $u_1, u_2 \in \Lambda$

those complex tori which admit a line bundle  $\mathcal{L} = \mathcal{L}(H, \alpha)$

s.t  $H$  is positive definite. Moreover, for every  $n \geq 3$ ,  $|\mathcal{L}^{\otimes n}|$  gives

a projective embedding of  $A(\mathbb{C}) \Rightarrow A$  is projective.

The projectivity of an abelian varieties over any field is more involved. See Mumford's AV.

## 1.2 Dual abelian varieties and polarizations.

Def  $A$ : abelian variety/ $k$ . The relative Picard functor

$\text{Pic}(A/k)$  is a group ( $k$ -scheme)  $\rightarrow$  (Group)

$$S \mapsto \left\{ (\mathcal{L}, \xi), \text{ where } \mathcal{L}: \text{line bundle on } A \times_k S, \xi: \mathcal{O}_S \xrightarrow{\sim} e^* \mathcal{L}, e: S \rightarrow A \right\} / \sim$$

Zero section

rigidified line bundle.



Thm/Def. The functor  $\text{Pic}(A/k)$  is representable by a smooth group scheme, denoted by  $\text{Pic}(A/k)$ , locally of finite type  $/k$ . ④

The neutral connected component  $\text{Pic}^0(A/k)$  is an abelian variety, called the **dual abelian variety** of  $A$ , denoted by  $A^t$ .

Let  $\mathcal{L} \in \text{Pic}(A)$ . Define  $\lambda_{\mathcal{L}}: A \rightarrow \text{Pic}(A/k)$ .

$T_x: A \rightarrow A$ , translation by  $x$ .  $x \mapsto T_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$   
 $y \mapsto x+y$ .

Then  $T_{x+y}^* \mathcal{L} \otimes \mathcal{L}^{-1} \simeq (T_x^* \mathcal{L} \otimes \mathcal{L}^{-1}) \otimes (T_y^* \mathcal{L} \otimes \mathcal{L}^{-1})$  (thm of square)

So  $\lambda_{\mathcal{L}}$  is a group homo.

Since  $A$  is connected,  $\lambda_{\mathcal{L}}$  factors thru:  $A \rightarrow A^t$ .

If  $f: A \rightarrow B$  gp homo, then  $f^t: B^t \rightarrow A^t$ : dual of  $f$   
 $\mathcal{L} \mapsto f^* \mathcal{L}$ .

There is a canonical isom  $\text{can}: A \xrightarrow{\sim} A^{tt}$ .

Via  $\text{can}$ :  $\lambda_{\mathcal{L}}^t: A \simeq (A^t)^t \rightarrow A^t$  and one has  $\lambda_{\mathcal{L}} = \lambda_{\mathcal{L}}^t$ .

If  $\mathcal{L}$  is ample, i.e.  $\mathcal{L}^{\otimes n}$ , for any  $n \gg 0$ , gives a projective embedding.

then  $\lambda_{\mathcal{L}}$  is an isogeny.

Def. (1) An isogeny  $\lambda: A \rightarrow A^t$  is called a **polarization** (5)

on  $A$  if  $\exists K/k$  finite sep field extn. and  $\mathcal{L}_K$  ample line bundle

$$\text{on } A_K = A \otimes K \text{ st } \lambda_{\mathcal{L}_K} = \lambda_K$$

The pair  $(A, \lambda)$  is called a **polarized abelian variety**.

If  $\lambda: A \xrightarrow{\sim} A^t$ , then  $\lambda$  is called **principal** and  $(A, \lambda)$

is called a **principally polarized abelian variety**.

(2) If  $\lambda: A \dashrightarrow A^t$  is a quasi-isogeny st.  $N \cdot \lambda$  for some  $N \in \mathbb{N}$

is a polarization, then  $\lambda$  is called a **fractional polarization** or

a  **$\mathbb{Q}$ -polarization**. That is,  $\mathbb{Q}_+ \cdot \lambda$  contains a polarization.

(do not confuse with the "weak polarization", which is  $\mathbb{Q}_+ \cdot \lambda$  or  $\mathbb{Q}^r \cdot \lambda$  of a polarization  $\lambda$ ).

Similarly an abelian variety  $A$  together with a  $\mathbb{Q}$ -polarization

is called a  **$\mathbb{Q}$ -polarized abelian variety**  $(A, \lambda)$ .

Typical way

we use this: if  $(A_1, \lambda_1)$  pol. AV.  $\varphi: A_1 \rightarrow A$  isogeny

then  $\exists ! \mathbb{Q}$ -polarization  $\lambda$  on  $A$  st  $\varphi^* \lambda = \lambda_1$ .

• A main reason we consider pol. AVs instead of AVs is

that we want to study families of AVs. The main difference:

$\text{Aut}(A, \lambda)$  is finite, while  $\text{Aut}(A)$  is infinite usually.

### 1.3 Rosati involution.

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$A/k$ ,  $\text{End}^\circ(A) = \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ . the endomorphism algebra of  $A/k$ .

If  $A \sim_k B$ , then  $\text{End}^\circ(A) \simeq \text{End}^\circ(B)$ .

Poincaré complete reducibility theorem:

$$A \sim_k B_1^{n_1} \times \cdots \times B_r^{n_r}, \quad n_i \geq 1, \quad B_i: k\text{-simple abel. var.}$$

$$\text{and } B_i \not\sim B_j \quad \forall i \neq j,$$

$$\Rightarrow \text{End}^\circ(A) \simeq \prod_{i=1}^r \text{Mat}_{n_i}(D_i), \quad \text{where } D_i = \text{End}^\circ(B_i)$$

fin-dim'l division alg /  $\mathbb{Q}$ .

Let  $\lambda$ : polarization on  $A$ ,  $\forall a \in \text{End}^\circ(A)$  define

$$a^* := \lambda^{-1} a^t \lambda$$

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & A^t \\ \downarrow a^* & & \downarrow a^t \\ A^t & \xleftarrow{\lambda^{-1}} & A \end{array} \quad \begin{array}{l} a: A \rightarrow A \\ a^t: A^t \rightarrow A^t \end{array}$$

$$\text{Then } (ab)^* = b^* a^*, \quad a^{**} = a,$$

$*$ : anti-involution, called the **Rosati involution** induced by  $\lambda$ .

Def. (1) Let  $B$  be fin-dim'l semi-simple  $\mathbb{Q}$ -algebra.

An involution  $*$  on  $B$  (i.e.  $*$ :  $B \xrightarrow{\sim} B^{\text{opp}}$  isom of order 2)

is said to be **positive** if  $\text{Tr}_{B/\mathbb{Q}} b \cdot b^* > 0 \quad \forall b \neq 0 \text{ in } B$ .

where  $\text{Tr}_{B/\mathbb{Q}}$  is the reduced trace from  $B$  to  $\mathbb{Q}$ .

(2) Let  $*$  be a positive involution on  $B$ , an element  $b \in B$  is said to be **symmetric** if  $b = b^*$ . ⑦

Put  $(x, y)_b := \text{Tr}_{B/\mathbb{Q}}(x \cdot b \cdot y^*) : B \times B \rightarrow \mathbb{Q}$ . Then

$$(b, x, y)_b = (x, b^* y)_b \quad \forall b, x, y \in B \quad \text{and}$$

$$\begin{aligned} (b, x, y)_b &= \text{Tr}(b, x \cdot b \cdot y^*) = \text{Tr}(x \cdot b \cdot y^* \cdot b) \\ &= \text{Tr}(x \cdot b \cdot (b^* y)^*) \\ &= (x, b^* y)_b. \end{aligned}$$

$(x, y)_b$  is symmetric  $\Leftrightarrow b$  is symmetric

(3) A symmetric element  $b$  is said to be **positive** if.

the symmetric pair  $(x, y)_b$  is positive definite.

Eg.  $\text{Mat}_n(\mathbb{R})$ ,  $t: A \rightarrow A^t$  positive  $x^t A y : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$   
 $\text{Mat}_n(\mathbb{C})$   $*$  :  $A \mapsto \bar{A}^t$  positive,  $a_{ij} \mapsto \bar{a}_{ji}$  complex conj  
 $\text{Mat}_n(\mathbb{H})$   $*$  :  $(a_{ij}) \mapsto (a_{ji}^*)$ ,  $a \mapsto a^*$  canonical involution  
 $\mathbb{H}$ : Hamilton quaternion /  $\mathbb{R}$ ,  $*$ ,  $a^* + a = \text{Tr}_{\mathbb{H}/\mathbb{R}} a$

gp of isometries:  $\{g \in B^* : g^* \cdot g = 1\} = \{g \in B^* : (gx, gy) = (x, y)\}$

$O(n), U(n), Sp_n^*$  compact inner

$$(x, y) = \text{Tr}_{B/\mathbb{R}}(x \cdot y^*)$$

form of  $Sp_{2n}(\mathbb{R})$   $\cong D$

Rmk.  $A$ : simple abelian var, then  $\text{End}^0(A)$ : fin-dim

division alg /  $\mathbb{Q}$  admitting a positive involution.

$(D, *)$  are classified Albert.  $\rightarrow$  Mumford's AV. Sec 21.

Thm (Riemann, Weil)  $(A, \lambda)$  polarized abel var /  $k$ .

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The Rosati involution on  $\text{End}^0(A)$  is positive.

Idea of Proof: /  $\mathbb{C}$ .  $*$  = the adjoint of the Riemann form

$$\langle, \rangle_\lambda = E : \Lambda \times \Lambda \rightarrow \mathbb{Z}$$

$$\text{End}^0(A) \subset H_1(A, \mathbb{Q})$$

$$E(ix, iy) = E(x, y) \quad x, y \in \Lambda_{\mathbb{R}}$$

$$\lambda \rightarrow \langle, \rangle_\lambda : H_1 \times H_1 \rightarrow \mathbb{Q}$$

non-deg alt. pairing

$a \in \text{End}^0(A)$   $a$  commutes with  $i$ .

pos. def.

$$* = \text{the adj. of the Hermitian form } H, \quad E = \text{Im } H$$

$\Rightarrow$   $*$  is positive.

/  $k$  of char 0. WMA  $k$ : fin. gen.  $k \subset \mathbb{C}$ .

/  $k$  of char  $p > 0$ , the algebraic proof is more complicated.

the proof uses the Riemann-Roch thm, intersection theory, and  $\ell$ -adic representations.

Cor  $\text{Aut}(A, \lambda)$  is finite.

$$\text{pf } G = \{ g \in \text{End}^0(A)_{\mathbb{R}}^{\times} : g^* g = 1 \} : \text{compact Lie group.}$$

$\cup$

$\uparrow$  discrete

$$\text{Aut}(A, \lambda) \subseteq \text{End}(A)^{\times} \text{ discrete subgp } \square.$$

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We mention a result of Norman-Oort, which relates char 0 & char  $p > 0$  abelian varieties. This completes a program of Mumford for which Mumford develops the theory of display.

**Thm** (Norman-Oort, 80) : Any polarized abelian variety  $(A, \lambda)$ .

over an alg closed field  $k$  of char  $p > 0$  can be lifted to char 0.

$\exists R \rightarrow k$  complete local Noetherian domain with  $K = \text{Frac}(R)$  of char 0.

and a polarized abelian

variety  $(A, \lambda) / R$  satisfying:

$$\begin{array}{ccc} (A, \lambda) & \hookrightarrow & (A, \lambda) \\ \downarrow & & \downarrow \\ \text{Spec } R & \hookrightarrow & \text{Spec } k \end{array}$$