3.3 Moduli spaces with parahoric level structure 06.12.2021.

$N \geq 3$, $(p, N) = 1$. $I = \{0, 1, \dots, g\}$

$\mathcal{A}_{g,I} = \mathcal{A}_I$: moduli space $/ \mathbb{Z}_{(p)}[\zeta_N]$ parameterizing complete chain of abelian varieties

$$(A_0 \xrightarrow{\alpha} A_1 \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} A_g, \lambda_0, \lambda_g, \eta),$$ where

- A_i : g-dim'l abelian variety.
- α : isogeny of degree p.
- λ_0, λ_g : principal polarizations on A_0, A_g, resp.
  s.t. $(\alpha^g)^* \lambda_g = p \cdot \lambda_0$.
- η : level-N structure on A_g.

$\pi : \mathcal{A}_I \longrightarrow \operatorname{Spec} \mathbb{Z}_{(p)}[\zeta_N]$: Siegel moduli scheme with Iwahori level structure (at p).

Thm (1) (Görtz, RZ conjecture) $\pi : \mathcal{A}_I \longrightarrow \operatorname{Spec} \mathbb{Z}_{(p)}[\zeta_N]$ is flat of relative dimension $g(g+1)/2$.

(2) (de Jong, Yu) The special fiber $\mathcal{A}_I \otimes \overline{\mathbb{F}}_p$ is connected and has 2^g irreducible components.

For each $\emptyset \neq J \subset I$, one associates a moduli space $\mathcal{A}_J$ of partial chain of abelian varieties indexed by J.

$$\left(A_{j_0} \xrightarrow{\alpha_1} A_{j_1} \to \cdots \xrightarrow{\alpha_r} A_{j_r},\ \lambda_{j_0}, \lambda_{j_r} \text{ polarizations o } A_{j_0}, A_{j_r}. \right.$$

s.t. · λ_{j_0} = pull-back of λ_{j_r} $\qquad J = \{j_0, \cdots, j_r\}$

· $\ker \lambda_{j_0} \subseteq A_{j_0}[p]$, (polarization of parahoric type).

$\Updownarrow$ polarization type of $\lambda_{j_0} = (p, \cdots, p, 1 \cdots 1)$.

$\mathcal{A}_J$ = Siegel moduli scheme with parahoric level structure of type J.

For example: $g = 2$. $I = \{0, 1, 2\}$. Iwahori

$$\begin{tikzcd}
 & I = \{0,1,2\} \arrow[dl] \arrow[dr] & \\
\text{Siegel parahoric } \{0,2\} \arrow[d] & & \{0,1\} \text{ Klingen parahoric} \arrow[dll] \arrow[d] \\
\text{hyperspecial } \{0\} & & \{1\} \text{ paramodular}
\end{tikzcd}$$

Thm: (1) (Görtz) $\pi: \mathcal{A}_J \to \operatorname{Spec} \mathbb{Z}_{(p)}[\zeta_N]$ flat of relative dim $g \cdot (g+1)/2$

(2) (Yu.) $\mathcal{A}_J \otimes \overline{\mathbb{F}}_p$ is connected and has $(k_1 + 1) \cdots (k_r + 1)$ irreducible components, where $k_i = j_i - j_{i-1}$.

In particular, if $|J| = 1$, then $\mathcal{A}_J \otimes \overline{\mathbb{F}}_p$ is irreducible.

One important ingredient of the proof for irreducibility the ordinary locus $\mathcal{A}_J^{ord} \subseteq \mathcal{A}_J$ is Zariski dense. So even for studying the geometry of the whole moduli space, we do need to study the classification of subvarieties.

§ 4 Dieudonné modules.

4.1. k : perfect field of char $p > 0$.

$W = W(k)$: the ring of Witt vectors over k, $B(k) = \mathrm{Frac}\, W$.

σ: Frobenius map on W and on $B(k)$.

Def. A Dieudonné module over k is a finite W-module M together a σ-linear map F (Frobenius) and a σ^{-1}-linear map V (Verschiebung) on M. s.t. $FV = VF = p$.

σ-linear : $F(am) = \sigma(a) \cdot F(m)$ $\quad a \in W,\ m \in M$

σ^{-1}-linear : $V(am) = \sigma^{-1}(a) V(m)$.

If M is free$/W$, it's equiv. to (M, F) s.t.

$$pM \subseteq FM \subseteq M \qquad (\because\ V = p \cdot F^{-1} : M \to M\ )$$

Examples (1) M : free W-basis e_1, e_2.

$$F: \begin{array}{l} e_1 \to e_1 \\ e_2 \to pe_2 \end{array} \qquad V: \begin{array}{l} e_1 \to pe_1 \\ e_2 \to e_2 \end{array} \qquad \text{ordinary}$$

(2) M free, W-basis e_1, e_2

$$F: \begin{array}{l} e_1 \to e_2 \\ e_2 \to pe_1 \end{array}, \qquad V: \begin{array}{l} e_1 \to e_2 \\ e_2 \to pe_1 \end{array} \qquad \text{supersingular.}$$

(3) General construction. $n = \operatorname{rank} M$

choose two W-bases $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, $1 \leq a \leq n$.

$$F: \begin{array}{l} x_1 \longmapsto p y_1 \\ \vdots \\ x_a \longmapsto p y_a \\ x_{a+1} \longmapsto y_{a+1} \\ \vdots \\ x_n \longmapsto y_n \end{array} \qquad V: \begin{array}{l} y_1 \longmapsto x_1 \\ \vdots \\ y_a \longmapsto x_a \\ y_{a+1} \longmapsto p x_{a+1} \\ \vdots \\ y_n \longmapsto p x_n . \end{array}$$

Conversely every W-free Dieudonné module arises in this way.

$$M/VM = k\langle x_{a+1}, \ldots, x_n \rangle, \quad VM/pM = k\langle x_1, \ldots, x_a \rangle.$$

$$[y_1, \ldots, y_n] = [x_1, \ldots, x_n] \begin{array}{c} \begin{array}{cc} a & n-a \end{array} \\ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \end{array}$$

$$\begin{aligned} F \cdot [x_1, \ldots, x_n] &= [py_1, \ldots, py_a, y_{a+1}, \ldots, y_n] \\ &= [y_1 \cdots y_n] \left[\begin{array}{c|c} p & \\ \hline & 1 \end{array}\right] = [x_1 \cdots x_n] \begin{bmatrix} A & B \\ C & D \end{bmatrix} \left[\begin{array}{c|c} p & \\ \hline & 1 \end{array}\right] \\ &= [x_1, \ldots, x_n] \begin{bmatrix} pA & B \\ pC & D \end{bmatrix} \end{aligned}$$

$$M \longleftrightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in GL_n(W)$$

$X_1, \ldots, X_a \in VM$ s.t $\bar{X}_1, \ldots, \bar{X}_a$ k-basis on VM/pM

$X_{a+1}, \ldots X_n \in M$ s.t $\bar{X}_{a+1}, \ldots, \bar{X}_n$ k-basis on M/VM.

$X_1', \ldots, X_n'$ another W-basis.

$$[X_1', \ldots, X_n'] = [X_1, \ldots, X_n] \cdot P, \qquad P = \begin{array}{cc} a & n-a \\ \left[\begin{matrix} A_1 & B_1 \\ pC_1 & D_1 \end{matrix}\right] & \end{array}$$

$$\begin{aligned} F \cdot [X_1', \ldots, X_n'] &= F \cdot [X_1, \ldots, X_n] \cdot \sigma(P) \\ &= [X_1, \ldots, X_n] \begin{bmatrix} pA & B \\ pC & D \end{bmatrix} \sigma(P) \\ &= [X_1', \ldots, X_n'] P^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \overbrace{\begin{bmatrix} p & \\ & 1 \end{bmatrix} \sigma(P) \begin{bmatrix} p^{-1} & \\ & 1 \end{bmatrix}}^{\sigma(P)^*} \begin{bmatrix} p & \\ & 1 \end{bmatrix} \end{aligned}$$

$$\Rightarrow \quad F \cdot [X_1', \ldots, X_n'] = [X_1', \ldots, X_n'] P^{-1} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \sigma(P)^* \begin{bmatrix} p & \\ & 1 \end{bmatrix}$$

$$= [X_1', \ldots, X_n'] \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}$$

Prop: M, M': Dieudonné modules with $a = \dim VM/pM = \dim VM'/pM'$.

Suppose M & M' are defined by $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $g' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}$, resp.

(1) $\quad M \simeq M' \iff \exists\, P = \begin{bmatrix} A_1 & B_1 \\ pC_1 & D_1 \end{bmatrix}$ s.t

$$g' = P^{-1} g \cdot \sigma(P)^*,$$

where $*$ is the involution $\begin{bmatrix} A_1 & B_1 \\ pC_1 & D_1 \end{bmatrix} \longmapsto \begin{bmatrix} A_1 & pB_1 \\ C_1 & D_1 \end{bmatrix}$

(2) Write $\bar{M} := M/pM$. Then $\bar{M} \simeq \bar{M}' \iff$

$$\exists \quad Q_1 = \begin{pmatrix} \bar{A}_1 & 0 \\ \bar{C}_1 & \bar{D}_1 \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} \bar{A}_2 & \bar{B}_2 \\ 0 & \bar{D}_2 \end{pmatrix} \quad \text{in } GL_n(k)$$

with $\bar{A}_1 \cdot \bar{A}_2 = id_a$, $\bar{D}_1 \cdot \bar{D}_2 = id_{n-a}$ s.t. $\bar{g}' = Q_2 \cdot \bar{g} \cdot \sigma(Q_1)$.

Pf: (1) follows from our computation.

(2) Write $P = \begin{pmatrix} A_1 & B_1 \\ pC_1 & D_1 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} A_2 & B_2 \\ pC_2 & D_2 \end{pmatrix}$

$$\begin{bmatrix} id_a & 0 \\ 0 & id_{n-a} \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ pC_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ pC_2 & D_2 \end{bmatrix} = \begin{bmatrix} A_1A_2 + pB_1C_2 & A_1B_2 + B_1D_2 \\ p(C_1A_2 + D_1C_2) & D_1D_2 + pC_1B_2 \end{bmatrix}$$

$\Rightarrow \quad \bar{A}_1 \cdot \bar{A}_2 = id_a, \quad \bar{D}_1 \cdot \bar{D}_2 = id_{n-a}$

$\bar{C}_1 \bar{A}_2 + \bar{D}_1 \cdot \bar{C}_2 = 0 \quad \Rightarrow \bar{C}_1$: can be arbitary $\because$ $\bar{C}_2$: solvable,

$\bar{A}_1 \cdot \bar{B}_2 + \bar{B}_1 \cdot \bar{D}_2 = 0 \quad \Rightarrow \bar{B}_2$ can be arbitary $\because$ $\bar{B}_1$ solvable.

$$g' = P^{-1} g \, \sigma(P)^* \Rightarrow \bar{g}' = \begin{pmatrix} \bar{A}_2 & \bar{B}_2 \\ 0 & \bar{D}_2 \end{pmatrix} \cdot \bar{g} \; \sigma\begin{pmatrix} \bar{A}_1 & 0 \\ \bar{C}_1 & \bar{D}_1 \end{pmatrix}$$

Put $Q_2 = \begin{pmatrix} \bar{A}_2 & \bar{B}_2 \\ 0 & \bar{D}_2 \end{pmatrix}$, $Q_1 = \begin{pmatrix} \bar{A}_1 & 0 \\ \bar{C}_2 & \bar{D}_1 \end{pmatrix}$ $\square$

4.2 Duality

Def. M: Dieudonné module over k.

$$M^t := \begin{cases} \mathrm{Hom}_W(M, W) & \text{if } M \text{: free as } W\text{-module.} \\ \mathrm{Hom}_W(M, B(k)/W) & \text{if } M \text{ of finite length. i.e. } p^n M = 0 \text{ for some } n. \end{cases}$$

and $Ff(m) := F(Vm)^\sigma$, $Vf(m) := f(Fm)^{\sigma^{-1}}$

for $f \in M^t$, $m \in M$.

Then M^t is a Dieudonne module $/k$, called the dual of M.

- $Ff(am) = f(V(am))^\sigma = [\sigma^{-1}(a) \cdot f(Vm)]^\sigma$

$$= a \cdot f(Vm)^\sigma = a \cdot (Ff)(m)$$

$F: M^t \to M^t$. same as V

- $F(af)(m) = [(af)(Vm)]^\sigma = \sigma(a) \cdot f(Vm)^\sigma = \sigma(a)(Ff)(m)$

$F: M^t \to M^t$ σ-linear. Similarly. $V: M^t \to M^t$ σ^{-1}-linear.

? Is there a uniform definition for M ?

$\because$ M: f.g. W-module & W complete, M is complete.

write $M = \varprojlim M/p^n M$. $\quad M^t = \varinjlim_n (M/p^n M)^t$

$$= \mathrm{Hom}(M, B(k)/W)$$

M^t not f.g. W-module, though good dual object.

4.3 p-divisible groups.

Def (1) A p-divisible group of height h over k is an inductive system $(G_n, G_n \hookrightarrow G_{n+1})$, or $G = \varinjlim G_n$.

(i) G_n : finite commutative group scheme of rank $p^{n \cdot h}$ $\forall n$

(ii) $G_n = G_{n+1}[p^n]$,

(iii) $p : G_{n+1} \to G_n$: faithfully flat.

(2) A p-divisible group over a base scheme S is an inductive system $(G_n, G_n \to G_{n+1})$ of locally free finite group schemes G_n satisfies (ii) and (iii).

$(G_n)_S$, $T \to S$ Then $(G_n)_T := (G_n \times_S T)_n$.

(3) G : finite comm. gp scheme / k.

Then we have a canonical filtration : $0 \subset G^m \subset G^0 \subset G$.

and G/G^0 is étale. G is called étale if $G^0 = 0$

multiplicative if G^D : étale i.e. $G = G^m$

unipotent if $G^m = 0$.

Def The Cartier dual G^D of G is the finite group scheme that represents the group functor $\mathcal{H}om(G, \mathbb{G}_m)$.

A : abelian variety $/k$. $A[p^\infty] = \varinjlim A[p^n]$: p-divisible gp

4.4 Classical Dieudonné theory

Thm k : perfect field of char $p>0$.

(1) There is an anti-equivalence of categories

$$\left(\begin{array}{l}\text{p-divisible}\\ \text{groups over } k\end{array}\right) \xrightarrow{M^*} \left(\begin{array}{l}\text{Dieudonné modules over}\\ k, \text{ free as } W\text{-modules}\end{array}\right)$$

(2) There is an anti-equivalence of categories

$$\left(\begin{array}{l}\text{commutative finite}\\ \text{group schemes over } k\end{array}\right) \xrightarrow{M^*} \left(\begin{array}{l}\text{Dieudonné modules over}\\ k \text{ of finite length}\end{array}\right)$$

(1) follows from (2) : $G = \varinjlim G_n$.

$M^*(G) := \varprojlim M^*(G_n)$, $M^*(G_n)$: free W/p^n - module of rank h.

For (2), one first constructs an anti-equiv of cat.

$$(*) \quad \left(\begin{array}{l}\text{comm. unipotent}\\ \text{finite gp schemes}/k\end{array}\right) \xrightarrow{M^*} \left(\begin{array}{l}\text{Dieudonné modules}/k\\ \text{of finite length, s.t } V\\ \text{is nilpotent}\end{array}\right)$$

and use the decompose $G = G^m \oplus G^{uni}$

If G multiplicative, $M^*(G) := M^*(G^D)^t$

To do (*), one first constructs a group scheme $\underrightarrow{W}$ over k, on which we have $W(k)$-module structure

and operators F, V which are σ-linear, σ^{-1}-linear s.t. $FV = VF = p$.

Then define $M^*(G) := \mathrm{Hom}(G, \underrightarrow{W})$.

An ad hoc covariant version : $M(G) = M^*(G)^t$.

for G : p-divisible group, or G : finite group scheme.

Remark

When G : connected p-divisible group $/k$

$\updownarrow$

p-divible $\underbrace{\text{formal group } / k}$

Cartier's Theory : $M_{Cart}(G)$.

$$M(G) \simeq M_{Cart}(G).$$

It is natural to extend the contravariant Dieudonné functor M^* to the category $\mathcal{C}$ which is the smallest abelian category in the abelian category of commutative group schemes/k (or viewed as fppf sheaves) that contains p-divisible groups and hence finite group schemes. Then there is an anti-equivalence of categories

$$M^* : \mathcal{C} \xrightarrow{\sim} \mathfrak{DM}_k = \begin{pmatrix} \text{Dieudonné modules} \\ \text{over } k \end{pmatrix}.$$

Then $0 \to H_1 \to H_2 \to H_3 \to 0$ exact seq in $\mathcal{C}$

$$\Leftrightarrow \quad 0 \to M^*(H_3) \to M^*(H_2) \to M^*(H_1) \to 0.$$

Question: M : Dieudonné module.

$$0 \to M_{tor} \to M \to M' \to 0, \quad M' : \text{free}/W$$

Does it split? Is there a canonical splitting.

This is useful for us to extend the duality as

$$M^t := M^t_{tor} \oplus (M')^t.$$

§5 Newton stratification.

5.1 Classification up to isogeny

$$F^m - V^n = 0$$
$$F^{m+n} = p^n$$
$$F \sim p^{\frac{n}{m+n}}$$

$(m,n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, $\gcd(m,n) = 1$.
$\neq 0$

$$M_{m,n} = W[F,V]/(F^m - V^n). \quad W\text{-module rank } m+n.$$

Then $M_{m,n}$ is a Dieudonné module, and call $\frac{n}{m+n}$: the slope of $M_{m,n}$ (with multiplicity $m+n$).

Thm (Dieudonné-Manin) M: any Dieudonné module / $k = \bar{k}$

$$\Rightarrow \quad M \otimes B(k) \simeq \bigoplus_{(m_i,n_i)} M_{m_i,n_i}^{r_i} \otimes B(k)$$

for some $0 \neq (m_i, n_i) \in \mathbb{Z}_{\geq 0}^2$, coprime pairs, and $r_i \geq 1$.

$=$

$$\text{slope}(M) := \Big\{ \underbrace{\Big(\frac{n_i}{m_i+n_i}\Big)}_{=\lambda_i}^{\oplus r_i(m_i+n_i)} \cdots \Big\}. \quad \lambda_i \leq \lambda_{i+1}$$

called the slope sequence of M. $\longleftrightarrow$ Newton polygon of M.

Note: the multiplicity of each slope λ_i is a multiple of its denominator.

integral condition.

$\leftrightarrow$ break points of NP are lattices.

If k is perfect, then $\text{slope}(M) := \text{slope}(M \otimes W(\bar{k}))$.

Def. (1) A Dieudonné module M over k is called supersingular if it has only slope $1/2$.

(2) The a-number of M is defined as

$$a(M) := \dim_k M/(F,V)M$$

$$a(M \otimes W(K)) = a(M) \quad \text{for any perfect extn } K/k.$$

(3) The p-rank of M is defined as

$$\text{p-rank}(M) = \operatorname{rank}_{\mathbb{Z}_p} M^{\acute{e}t}, \quad \text{where } M^{et} = \{x \in M \otimes W(\bar{k}) : \underline{Fx = x}\}.$$

If $M = M^*(G)$, G: p-divisible gp then $\text{p-rank}(G) = f$.

where f = the integer s.t. $G[p](\bar{k}) = (\mathbb{Z}/p)^f$.

(4) A: abelian variety $/ k = \bar{k}$, $M = M^*(A[p^\infty])$ = the (contr.) Dieudonné module of A. Define

$\text{slope}(A) := \text{slope}(M)$, A is supersingular if M is supersingular.

$a(A) := a(M)$, A is ordinary if $f\text{-rank}(M) = g$, $g = \dim A$

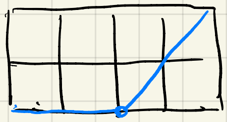
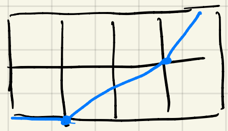
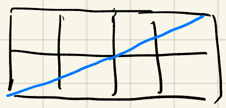
$\text{p-rank}(A) := \text{p-rank}(M)$, A is almost ordinary if $f = g-1$

A is superspecial if $a(A) = g$.

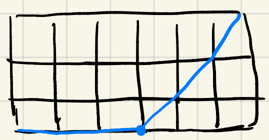
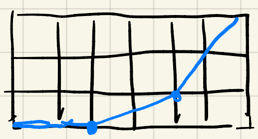
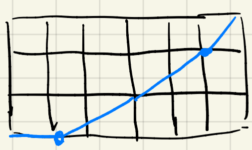
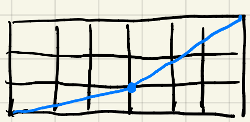
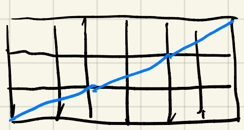
Note: p-rank is an isogeny invariant

a-# is not an isogeny invariant.

$g = \dim(A) = 2$

			f	a-#	dim
$0\ 0\ 1\ 1$		ordinary	2	0	3
$0\ \frac{1}{2}\ \frac{1}{2}\ 1$		almost ordinary	1	1	2
$\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}$		supersingular	0	1, 2 (2: superspecial)	1

$g = 3$: 5 Newton polygons

			p-rank	a-#	dim
$0\ 0\ 0\ 1\ 1\ 1$		ordinary	3	0	6
$0\ 0\ \frac{1}{2}\ \frac{1}{2}\ 1\ 1$		almost ordinary	2	1	5
$0\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ 1$			1	1, 2 (2: ord × ssp)	4
$\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}\ \frac{2}{3}\ \frac{2}{3}\ \frac{2}{3}$			0	1, 2	3
$\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}$		supersingular	0	1, 2, 3 (3: superspecial)	2

discussions:

M: étale Dieudonné module / k: perfect. slope 0.

$$\updownarrow$$

$\rho: \mathrm{Gal}(\bar{k}/k) \longrightarrow GL_n(\mathbb{Z}_p)$ contiuous Galois representation.

$M \otimes W(\bar{k}) \simeq M_{1,0}^{r} \otimes W(\bar{k})$, for some r

$$G[p]_{\bar{k}} = \mu_p^{r} \times G[p]^{loc.loc.} \times \underbrace{G[p]^{\text{ét}}}_{= (\mathbb{Z}/p)^{\boxed{f}}} \quad \text{general.}$$

$$G[p](\bar{k}) = (\mathbb{Z}/p)^{f}.$$

$M^{\text{ét}}$: Dieudonné module / $\mathbb{F}_p$.

$$\cong M_{1,0}^{\oplus f} \qquad M_{1,0} = W(\mathbb{F}_p)[F,V]/(F-1)$$

$$= \mathbb{Z}_p[F,V]/(F-1)$$

(so $V=p$)

$$a(A) = \dim \mathrm{Hom}(\alpha_p, A)$$

$$a(G) = \dim \mathrm{Hom}(\alpha_p, G)$$

$$M^*(\alpha_p) = k \quad , \quad F = V = 0$$

$$\mathrm{Hom}_{gp}(\alpha_p, G) = \mathrm{Hom}_{DM}\Big(\overset{M}{\overset{\shortparallel}{M^*(G)}}, \overset{k}{\overset{\shortparallel}{M^*(\alpha_p)}}\Big)$$

$$= \mathrm{Hom}_{\substack{W \\ \shortparallel\ k}}\big(M/(F,V)M, \ k\big)$$

$$\mathrm{Hom}_{gp}(\alpha_p, G) \underset{can}{\cong} \underbrace{\big(M/(F,V)M \big)^*}_{k\text{-v.s.}}$$