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Prop 5 : ϕ : Drinfeld A-module / L, $L = \bar{L}$: alg closed field. of rank r and height h

(1) If $0 \neq I \subset A$ is prime to $\mathrm{char}_A L$, then $\phi[I](L) \simeq (A/I)^r$.

(2) Suppose $\mathfrak{p} = \mathrm{char}_A L$ is finite, Then $\forall e \in \mathbb{N}$

$$\phi[\mathfrak{p}^e](L) \simeq (A/\mathfrak{p}^e)^{r-h}.$$

Pf : First, $\phi(L) = L$ with A-module str. given by

$$a \cdot \ell = \phi_a(\ell) \qquad \ell \in L,\ a \in A.$$

By definition $\quad \phi[I](L) = \phi(L)[I]$

$\because L = \bar{L}$, $\forall a \neq 0 \in A$, $\phi_a : L \to L$: surjective.

$\Rightarrow \phi(L)$: A-divisible.

By Lemma : it suffices to show for $I = \mathfrak{p}^e$, $\mathfrak{p}$: prime ideal.

and that $\exists e \in \mathbb{N}$ s.t

$$\phi(L)[\mathfrak{p}^e] \simeq (A/\mathfrak{p}^e)^r \qquad \text{if } \mathfrak{p} \neq \mathrm{char}_A L.$$

$$\phi(L)[\mathfrak{p}^e] \simeq (A/\mathfrak{p}^e)^{r-h} \qquad \text{if } \mathfrak{p} = \mathrm{char}_A L.$$

(so that the above holds for ALL $e \geq 1$)

Let $e \in \mathbb{N}$ s.t $\mathfrak{p}^e = (a)$, $a \in A$. $\qquad \deg(a) = e \cdot \deg(\mathfrak{p})$

$\deg \phi_a = r \cdot \deg(a) = r \cdot e \cdot \deg(\mathfrak{p})$, $\quad |A/\mathfrak{p}^e| = q^{e \cdot \deg(\mathfrak{p})}$

(1) If $\mathfrak{p} \neq \mathrm{char}_A L$, then $|\phi(L)[\mathfrak{p}^e]| = q^{r \cdot \deg(a)} = q^{r \cdot e \cdot \deg(\mathfrak{p})}$

$$\phi[\mathfrak{p}^e](L) \simeq (A/\mathfrak{p}^e)^r.$$

(2) If $\mathfrak{p} = \mathrm{char}_A L = v$, $\phi_a(z) = \underset{\neq 0}{a_w} z^{q^{w(\phi_a)}} + \cdots + \underset{\neq 0}{a_*} z^{q^{r\cdot\deg(a)}}$

$$|\phi[a](L)| = q^{r\cdot\deg(a) - w(\phi_a)} = q^{e\cdot\deg(v)(r-h)}$$

$r\cdot\deg(a) = r\cdot e\cdot\deg(v)$

$w(\phi_a) = h\cdot v(a)\deg(v) = h\cdot e\cdot\deg(v)$

$\Rightarrow \quad \phi[a](L) \simeq (A/a)^{r-h}. \qquad (a) = \mathfrak{p}^e$

Isogenies :

Def : A morphism $f : \phi_1 \to \phi_2$ of Drinfeld A-module / L.

is called an *isogeny* if $f \neq 0$.

Prop 6 : Isogenous Drinfeld modules have the same rank and height.

Pf : $f \neq 0, \in \mathrm{Hom}(\phi_1, \phi_2)$, $\quad f \circ \phi_{1,a} = \phi_{2,a} \circ f$.

$\Rightarrow \deg(\phi_{2,a}) + \deg(f) = \deg(f) + \deg(\phi_{2,a})$

$\Rightarrow \deg(\phi_{i,a}) = r_i \cdot \deg(a) \quad \Rightarrow \quad r_1 = r_2$

Similarly $h_1 = h_2$.

(II) Families of Drinfeld modules.

k : any commutive ring of char p. $\mathrm{End}(\mathbb{G}_{a,k}) = k\{\tau\}$. $\tau(z) = z^p$.

$\mathrm{Aut}(\mathbb{G}_{a,k}) \cong k\{\tau\}^\times$.

$$k\{\tau\} \supset k\{\tau_q\} \quad \tau_q(z) = z^q.$$

Lemma 1: An element $\sum_{i=0}^{n} \alpha_i \tau^i \in k\{\tau\}$ is invertible

$\Leftrightarrow$ $\alpha_0 \in k^\times$ and $\alpha_1, \dots, \alpha_n$: nilpotent.

Pf: Exercise.

Lemma 2: $d \in \mathbb{N}$. $\varphi = \sum \alpha_n \tau^n$, $\psi = \sum \beta_n \tau^n \in k\{\tau\}$.

s.t. $\alpha_d, \beta_0 \in k^\times$ and α_n $(n > d)$ and β_n $(n > 0)$ are nilpotent.

Set $\psi \circ \varphi \circ \psi^{-1} = \sum \delta_n \tau^n$

Then (i) $\delta_d \in k^\times$ and for $n > d$, δ_n is nilpotent

(ii) $\exists ! \psi$ s.t. $\beta_0 = 1$ and $\delta_n = 0$ $\forall n > d$.

Pf: (i) Write $\psi^{-1} = \sum \gamma_n \tau^n$. so $\gamma_0 \in k^\times$, γ_n : nilpotent $\forall n > 0$.

Then $$\delta_n = \sum_{i+j+k=n} \beta_j \alpha_i^{p^j} \cdot \gamma_k^{p^{i+j}}.$$

$\beta_j \cdot \alpha_i^{p^j} \cdot \gamma_k^{i+j}$: nilpotent if $j > 0$, $i > d$, $k > 0$.

$\Rightarrow$ For $n > d$, δ_n : nilpotent

$n = d$, $$\delta_d = \underbrace{\beta_0 \alpha_d \gamma_0^{p^d}}_{\substack{(i=d) \\ \text{unit}}} + \underbrace{\sum_{\substack{j>0 \\ \text{or } k>0}} \beta_j \alpha_i \gamma_k^{p^{i+j}}}_{\substack{(i<d) \\ \text{nilpotent}}} \Rightarrow \delta_d : \text{invertible}$$

(ii) Existence : $I = (\alpha_n : n > d)$: nilpotent ideal of A.

$\exists s$ s.t. $I^s = 0$. First, we may assume $I^2 = 0$.

$$\left(\begin{array}{l} \exists \psi_1 \quad \varphi_1 = \psi_1 \cdot \varphi \cdot \psi_1^{-1} \equiv \sum^d \delta_n^{(1)} \tau^n \mod I^2 \\ \exists \psi_2 \quad \varphi_2 = \psi_2 \varphi_1 \psi_1^{-1} \equiv \sum^d \delta_n^{(2)} \tau^n \mod I^4 \\ \quad \vdots \\ \varphi_r \equiv \psi_r \varphi_{r-1} \psi_r^{-1} \equiv \sum^d \delta_n^{(r)} \tau^n \mod I^{2^r} = 0 \end{array} \quad \left(\underline{k/I^2},\ \overset{\overline{I}}{\overset{\shortparallel}{I/I^2}} \right) \right)$$

$$\varphi = \sum_{i=0}^{N} \alpha_i \tau^i \qquad N > d, \qquad \exists \psi, \quad \psi \cdot \varphi \psi^{-1} = \sum^{N-1} \alpha_i' \tau^i$$

Take $\psi = (1 - \beta \tau^{N-d})$ $\beta^2 = 0$, $\psi^{-1} = (1 + \beta \tau^{N-d})$

$$\psi \varphi \psi^{-1} = (1 - \beta \tau^{N-d})(\alpha_0 + \cdots + \alpha_d \tau^d + \cdots + \alpha_N \tau^N)(1 + \beta \tau^{N-d})$$

$$= (\alpha_0 + \alpha_d \tau^d + \cdots + \underline{\alpha_N \tau^N} + \cdots + \underline{\beta \cdot \alpha_d^{p^{N-d}}} \tau^N)(1 + \beta \tau^{N-d}), \quad \tau\beta = 0$$

$$= (\quad \cdots + (\alpha_N + \beta \alpha_d^{p^{N-d}}) \tau^N). \quad \text{put } \beta = -\alpha_N / \alpha_d^{p^{N-d}}.$$

$(\beta^2 = 0)$

<u>Uniqueness</u>: Suffices to show if

$$\varphi = \sum^d \alpha_i \tau^i, \quad \psi \varphi \psi^{-1} = \sum^d \delta_n \tau^n, \text{ then } \psi = 1.$$

$$\left(\sum^N \beta_j \tau^j \right) \left(\sum^d \alpha_i \tau^i \right) = \left(\sum^d \delta_n \tau^n \right) \left(\sum^N \beta_j \tau^j \right)$$

$$\tau^{N+d} : \quad \beta_N \alpha_d^{p^N} = \delta_d \cdot \beta_N^{p^d} \qquad \beta_N = u \cdot \beta_N^{p^d} = \cdots = 0$$

$\Rightarrow \psi = 1$.

Families of Drinfeld modules

S : base scheme of char p. i.e. $S \to \operatorname{Spec} \mathbb{F}_p$.

$d \in \mathbb{N}$.

Def : (1) A Drinfeld A-module of rank d/S is a pair (E, φ),

where E : commutative group scheme $/S$, and

$\varphi : A \to \operatorname{End}_{gp}(E)$: ring homomorphism s.t

(i) $\forall x \in S$ $\exists$ affine open subset $U \ni x$ and an isom

$$\psi : E|_U \xrightarrow{\sim} \mathbb{G}_{a,U} \text{ of gp schemes } / U.$$

(ii) If $U = \operatorname{Spec} k \subset S$ open affine subscheme and ψ : isom as above.

then $\forall a \neq 0 \in A$, let

$$\psi \circ \varphi(a) \circ \psi^{-1} = \sum \alpha_n(a) \tau^n \in k\{\tau\},$$

the following are satisfied :

- $\alpha_n(a) \in k^\times$ for $n = d \cdot \deg(a)$
- $\alpha_n(a)$: nilpotent for $n > d \cdot \deg(a)$.

Two Drinfeld A-modules (E_1, φ_1), (E_2, φ_2) are isomorphic

if $\exists$ $\alpha : E_1 \to E_2$ isom of gp schemes which is compatible

with the actions of A.

(2) $r \in \mathbb{N}$. let $\mathbb{A}^r_S$ denote the scheme of $\mathcal{O}_S$-modules def'd by

$\forall T \to S$ over S $\quad \mathbb{A}^r_S(T) = \Gamma(T, \mathcal{O}_T)^r$. free $\Gamma(T, \mathcal{O}_T)$-module of rank r.

called the trivial vector bundle of rank r.

A vector bundle of rank r is a scheme of $\mathcal{O}_S$-modules

s.t. $\forall x \in S$, $\exists$ affine open subset $U \ni x$ and an isom

$$\varphi : E_U \simeq \mathbb{A}^r_U \text{ of schemes of } \mathcal{O}_U\text{-module}.$$

If $r=1$, then E is called a (geom) line bundle over S.

(3) A standard Drinfield A-module of rank d / S is a pair (E, φ).

where $\cdot E$: line bundle over S, and

$\cdot\ \varphi : A \to \mathrm{End}_{gp}(E)$ ring homo. satisfying

$\cdot\ \forall a \neq 0 \in A$. $\quad \varphi(a) = \sum_{n=0}^{d.\deg(a)} \alpha_n(a)\, \tau^n$.

where $\tau^n : E \to E^{\otimes p^n}$ nth power of Frobenius morphism.

$\qquad s \mapsto s^{p^n}$

$\alpha_n(a) \in H^0(S, E^{\otimes(1-p^n)})$. $\qquad E \xrightarrow{\tau^n} E^{\otimes p^n} \xrightarrow{\alpha_n(a)} E$

$H^0(S, \mathcal{O}_S)$

$E \simeq \mathbb{A}^1 \simeq \mathbb{G}_a$

s.t. $\quad \alpha_{d\cdot\deg(a)}(a)$: never vanishing on S $\quad \forall a \neq 0 \in A$

$(\ \alpha_n(a) = 0 \quad \forall n > d \cdot \deg(a)\ )$

Every standard Drinfeld A-module $/ S \rightsquigarrow$ Drinfeld A-modules $/S$ by forgetting the $\mathcal{O}_S$-module structure.

Prop 3: Every Drinfeld A-module $/ S$ is isomorphic to (one arsing from) a standard Drinfeld A-module $/ S$ of the same rank.

Pf: This can be proved using Lemma 2, skip details.

$$\left(\begin{array}{c}\text{Standard Drinfeld} \\ A\text{-modules of rank} \\ d / S\end{array}\right) \xrightarrow[\text{fully faithful}]{\text{nat}} \left(\begin{array}{c}\text{Drinfeld } A\text{-modules} \\ \text{of rank } d / S\end{array}\right)$$

Prop 3: the essential image of this functor = everything.

$\Rightarrow$ they are equivalene of categories.

(E, φ): Drinfeld A-module $/ S$. Then the derivative

$$\partial : \mathrm{End}_{gp}(E) \longrightarrow H^0(S, \mathcal{O}_S), \quad \text{ring homo.}$$

($f : E \to E$, $df : \mathrm{Lie}(E) \longrightarrow \mathrm{Lie}(E)$, $\quad$ $\mathrm{Lie}(E)$ loc. free rank 1

$\mathcal{O}_S \to \mathrm{Lie}(E) \otimes \mathrm{Lie}(E)^{\vee}$

$df \in H^0(S, \mathcal{O}_S)$, $\quad \partial : f \mapsto df$)

The characteristic of (E, φ) is the ring homo

$$\partial \circ \varphi : A \to \mathrm{End}_{gp}(E) \xrightarrow{\partial} H^0(S, \mathcal{O}_S)$$

or $\quad \mathrm{Spec}(\partial \circ \varphi) : S \to \mathrm{Spec}\, A$.

Note: X: scheme, A: commutative ring. There is a canonical isom

$$\mathrm{Hom}(X, \mathrm{Spec}\, A) \simeq \mathrm{Hom}(A, \Gamma(X, \mathcal{O}_X))$$

$$f \longmapsto f^{\#} \text{ the pull-back operator}$$

$$a \longmapsto f^* a$$

Etale morphisms (Milne: Étale cohomology).

We choose

Settings: $\boxed{(a)}$ X: locally Noetherian scheme,

$f: Y \to X$ morphism locally of finite type.

(b) X: any scheme, $f: Y \to X$ morphism locally of finite presentation.

Def: $f: Y \to X$ morphism of schemes, loc. of finite type.

$y \in Y$, $x = f(y) \in X$.

(1) f is said to be unramified at y if (i) $m_x \mathcal{O}_{Y,y} = m_y$.

(ii) $k(y)$ is a finite separable extn of $k(x)$.

(2) If f is unramified at all $y \in Y$, then f is called unramified.

(3) f is said to be étale at y if f is flat & unramified at y.

(4) If f is étale at all $y \in Y$, then f is called étale.

$f: Y \to X$ morphism of schemes, loc. of f.t. Assume f: separated, i.e.

$$\Delta: Y \to Y \times_X Y \text{ closed immersion}, \quad Y \underset{\sim}{\xrightarrow{\Delta}} \Delta(Y).$$

$\mathcal{I}_Y$ = sheaf of ideals of $Y \times_X Y$ defining $\Delta(Y)$.

Define $\Omega^1_{Y/X} := \Delta^*(\mathcal{I}_Y/\mathcal{I}_Y^2)$, the sheaf of Kähler differentials.

Prop 4 : $f: Y \to X$ as above, TFAE:

(a) f: unramified,

(b) $\Omega^1_{Y/X} = 0$,

(c) $\Delta: Y \to Y \times_X Y$ open immersion.

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level structure :

$I \neq 0 \subset A$ ideal (E, φ): Drinfeld A-module of rank d / S.

$V(I) \subset X \setminus \{\infty\} = \operatorname{Spec} A$: closed subscheme def'd by I

$\quad \| $

$\operatorname{Spec} A/I$. $\quad \theta: S \to \operatorname{Spec} A$ char of (E, S).

The I-torsion subgp scheme of (E, φ) is the subscheme of E:

$$E_I = \bigcap_{a \in I} \left(\ker \varphi(a): E \to E\right)$$

Clearly E_I : sub A-module of E,

If $I = I_1 \cdot I_2$, $I_1 + I_2 = A$, $A/I = A/I_1 \times A/I_2$ & $E_I = E_{I_1} \times_S E_{I_2}$.

Prop 5: Assume that $\theta : S \longrightarrow \mathrm{Spec}\, A \setminus V(I) = \mathrm{Spec}\, A[I^{-1}]$. Then E_I is locally constant with value isomorphic to $(A/I)^d$ for the étale topology on S.

Pf: Need to show $\forall\, s \in S$ $\exists$ connected étale nbd U of s.

s.t. $E_I(U) \simeq (A/I)^d$.

First, choose $J \neq 0 \subseteq A$ ideal s.t. (i) $\theta(s) \notin V(J)$, (ii) $I \cdot J = (a)$

(iii) $I + J = A$.

To do this, first choose J s.t. $I \cdot J = (a)$

use weak approximation $\exists\, b \in F^\times$ s.t. $J \cdot b$ prime to $\theta(s)$ & I

replace J by $J b$.

So $\theta(a) \notin V(a)$. Suppose the prop holds for $I = (a)$ $(*)$

Then $\exists$ conn. étale nbd U of s. s.t

$$E_a(U) \simeq (A/a)^d = (A/I)^d \times (A/J)^d.$$

On the other hand, $E_{I \cdot J}(U) = E_I(U) \times E_J(U)$

$$\Rightarrow \quad E_I(U) \simeq (A/I)^d.$$

To prove $(*)$, We may assume $S = \mathrm{Spec}\, R$, $\theta : A[a^{-1}] \to R$, $\theta(a) \in R^\times$

$\varphi(a)[z] = \theta(a) z + a_1 z^q + \cdots + a_r z^{q^r}$ étale polynomial.

E_a: finite étale cover of S of constant rank $q^{d \cdot \deg(a)}$

E_{a^2} = $q^{d \cdot \deg(a^2)} = d \cdot \deg(a) \cdot 2$

$M = E_{a^2}(U)$, $M_a = E_a(U) \simeq (A/a)^r$

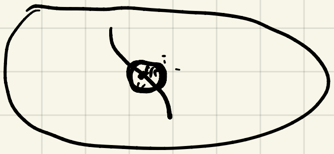
Lemma 6: M: finite A/a^2-module, $M_a = \ker(a : M \to M)$.

Then (i) $\# M \leq (\# M_a)^2$
$=$

(ii) "=" in (i) holds $\iff$ M: free A/a^2-module of finite rank.

$(\Rightarrow M_a = (A/a)^r)$

Pf: Exercise.



$1 \leq h \leq d$

$S = \bigsqcup_{1 \leq h \leq r} S^{(h)}$