

## § 7 Ekedahl-Oort stratification.

## 7.1. EO types.

$k = \bar{k}$ , of char  $p > 0$ ,  $g \geq 1$ .

$$\mathcal{BT}_1(g) := \left\{ \begin{array}{l} \text{isom classes of p.p.} \\ \mathcal{BT}_1(G, \lambda) \text{ over } k \\ \text{of rank } p^{2g} \end{array} \right\}$$

$$\mathcal{DM}_1(g) := \left\{ \begin{array}{l} \text{isom classes of p.p.} \\ \mathcal{DM}_1(\bar{M}, \langle, \rangle) \text{ over } k \\ \text{of rank } 2g. \end{array} \right\}$$

$$(G_0, \lambda_0) \in \mathcal{BT}_1(g)$$

$$S_{(G_0, \lambda_0)} := \left\{ (A, \lambda) \in \mathcal{A}_g(k) : (A, \lambda)[P] \simeq (G_0, \lambda_0) \right\}.$$

: locally closed subset of  $\mathcal{A}_g$ , regarded as

a reduced subscheme of  $\mathcal{A}_g$ .

called the **EO stratum** ass to  $(G_0, \lambda_0)$ .

$$\mathcal{A}_g = \bigsqcup_{(G_0, \lambda_0) \in \mathcal{BT}_1(g)} S_{(G_0, \lambda_0)} : \text{EO stratification.}$$

Def. (1) An elementary sequence of length  $g$  is a map

$$\varphi: \{0, 1, \dots, g\} \rightarrow \{0, 1, \dots, g\} \text{ s.t.}$$

$$\varphi(0) = 0, \text{ and } \varphi(i-1) \leq \varphi(i) \leq \varphi(i-1) + 1, \quad \forall 1 \leq i \leq g.$$

Denote by  $ES$  the set of elementary sequences of length  $g$ .

Also write  $(\varphi(1), \dots, \varphi(g))$  for  $\varphi$ .

(2) A final sequence of length  $g$  is a map

$$\psi: \{0, 1, \dots, 2g\} \rightarrow \{0, 1, \dots, g\} \text{ with } \psi(0) = 0, \psi(2g) = g.$$

$$\text{s.t. (i) } \psi(i-1) \leq \psi(i) \leq \psi(i-1) + 1 \quad i = 1, 2, \dots, 2g.$$

$$(ii) \quad \psi(i) < \psi(i+1) \text{ occurs} \iff \psi(2g-i) = \psi(2g-i-1) \text{ occurs}$$

$$(iii) \quad \psi(2g-i) = g-i + \psi(i), \quad \forall i.$$

Denote by  $FS$  the set of final sequences of length  $g$ .

The map  $FS \xrightarrow{\sim} ES$ .

$$\psi \mapsto \varphi = \psi|_{\{0, 1, \dots, g\}}$$

e.g.  $g=4, \quad \varphi = 0112$

	0	1	2	3	4	5	6	7	8
$\varphi$	0	0	1	1	2	2	3	3	4
			+		+		+		+

$g$ : midpoint

$$\psi(g+i) - \psi(g-i) = i, \quad i = 1, 2, \dots, g.$$

Thm (Kraft, Dort) There is a bijection between  $ES$  and  $\mathcal{BT}_g$ .

If  $\varphi \in ES$  and  $(G, \chi) \in BT_1$ , the corresponding  $BT_1$ , then

(56)

define  $S_\varphi = S(G, \chi)$ , called the **EO stratum** ass to  $\varphi$ .

**7.2** Construction of the correspondence from  $(\bar{M}, \langle, \rangle)$  to  $\Psi$ .

$(\bar{M}, \langle, \rangle)$ : p.p.  $DM_1$  of rank  $2g$ .

For each word  $w$  in  $\{F, \perp\}$ , denote by  $w(\bar{M})$  the subspace obtained by applying operations in  $w$  on  $\bar{M}$ .

e.g.  $w = F \cdot \perp \cdot F$ ,  $w(\bar{M}) = F((F\bar{M})^\perp)$ .

Consider  $\{w \cdot \bar{M} : \bar{w} : \text{any word}\}$

**Prop.** This is a filtration of subspaces

$$W. \quad 0 = W_0 \subsetneq W_1 \subseteq \dots \subseteq W_r \subseteq \dots \subseteq W_{2r} = \bar{M}$$

satisfying  $\begin{matrix} W_r = F \cdot \bar{M} \\ \Rightarrow W_r : \text{max. isotropic} \end{matrix}$

$$(i) \quad \forall 0 \leq j \leq 2r, \quad W_j^\perp = W_{2r-j} \quad (\text{symmetric})$$

$$(ii) \quad \exists \text{ surjective map } \nu : \{0, 1, \dots, 2r\} \rightarrow \{0, \dots, r\}$$

$$\text{s.t. } F(W_j) = W_{\nu(j)}$$

$W.$  the **canonical filtration** of  $(\bar{M}, \langle, \rangle)$ .

$P: \{0, \dots, 2r\} \rightarrow \mathbb{N}$  s.t.  $P(i) = \dim W_i$ . P. determined by  $P(1), \dots, P(r)$

$$P(i) + P(2r-i) = 2g$$

$0 = P(0) < \underbrace{P(1) < \dots < P(r)}_g < P(2r) = 2g$ ,  $\therefore$  increasing seq from 0 to  $2g$  in  $2r$  steps.

(57)

Define a map  $\Psi: \{0, 1, \dots, 2g\} \rightarrow \{0, 1, 2, \dots, g\}$  with  $\Psi(0) = 0$

as follows: Suppose  $\Psi(0), \dots, \Psi(P(i))$  are defined.

For  $P(i)+1 \leq j \leq P(i+1)$ , define

$$\Psi(j) := \begin{cases} \Psi(j-1) & \text{if } v(i+1) = v(i) \\ \Psi(j-1) + 1 & \text{if } v(i+1) > v(i) \end{cases}$$

Assertion  $\Psi \in FS$ , = the final seq of  $(\bar{M}, \langle, \rangle)$ .

Example 5 (1)  $(\bar{M}, \langle, \rangle)$ : superspecial,

$$0 \subset W_1 \subset W_2 = \bar{M}$$

$\nwarrow \quad \nwarrow$   
 $F \quad F$

$$F \cdot W_2 = W_{v(i)}$$

$$v: \{0, 1, 2\} \rightarrow \{0, 1\}$$

$$\begin{array}{ccc} 2 & \rightarrow & 1 \\ & & \# \text{ ②} \\ 1 & \rightarrow & 0 \\ & & \# \text{ ①} \\ 0 & \rightarrow & 0 \end{array}$$

$$\begin{array}{ccccc} & \text{①} & & \text{②} & \\ \Psi(0) & \xrightarrow{\quad} & \Psi(g) & \xrightarrow{\quad} & \Psi(2g) \\ \parallel & & \parallel & & \parallel \\ P(0) & & P(1) & & P(2) \end{array}$$

$$\underbrace{0 \ 0 \ \dots \ 0}_{\Psi} \ 1 \ 2 \ \dots \ g$$



(2)  $(\bar{M}, \langle, \rangle)$  : ordinary

(58)

$M$   $F: e_i \mapsto e_i$   $\langle e_i, e_{g+i} \rangle = - \langle e_{g+i}, e_i \rangle = 1$   
 $e_{g+i} \mapsto p \cdot e_{g+i}$  other pairings = 0

$\{e_1, \dots, e_g\}$

$0 \leq W_1 \leq W_2 = \bar{M}$

$v: \{0, 1, 2\} \rightarrow \{0, 1\}$

$\begin{matrix} p(0) & & p(1) \\ \parallel & \textcircled{1} & \parallel \\ \psi(0) & \xrightarrow{\quad} & \psi(1) \end{matrix}$

$\begin{matrix} p(2) & & 1 \\ \parallel & & \parallel \\ \psi(2) & \xrightarrow{\quad} & 1 \\ & & \textcircled{2} \\ & & \parallel \\ & & 0 \\ & & \textcircled{1} \end{matrix}$

$0 \ 1 \ 2 \ \dots \ g \ g \ \dots \ g$   
 $\underbrace{\hspace{10em}}_{\varphi}$

e.g.

ord

s.s.

(3) ordinary  $\oplus$  superspecial,  $g=2$ ,  $(\bar{M}, \langle, \rangle) = (\bar{M}_1, \langle, \rangle) \oplus (\bar{M}_2, \langle, \rangle)$

$\bar{M}_1$

$\bar{M}_2$

$F: e_1 \mapsto e_1$   
 $e_2 \mapsto p e_2$

$e_3 \mapsto e_4$   
 $e_4 \mapsto -p e_3$

$\langle e_1, e_2 \rangle = \langle e_3, e_4 \rangle = 1$

$0 \leq V_1 \leq \bar{M}_1$   
 $\underbrace{\hspace{2em}}_{F} \underbrace{\hspace{2em}}_{F}$

$0 \leq V_2 \leq \bar{M}_2$   
 $\underbrace{\hspace{2em}}_{F} \underbrace{\hspace{2em}}_{F}$

$v: \{0, 1, 2, 3, 4\} \rightarrow \{0, 1, 2\}$

$FW_i = W_{v(i)}$

$V_1^\perp$

$0 \leq V_1 \leq V_1 \oplus V_2 \leq V_1 \oplus \bar{M}_2 \leq \bar{M}$   
 $\parallel \quad \parallel \quad \parallel$   
 $W_1 \quad W_2 \quad W_3$

$\psi(0) \ \psi(1) \ \psi(2) \ \psi(3) \ \psi(4)$

$0 \quad 1 \quad 1 \quad 2 \quad 2$

$\varphi = 11$

$\begin{matrix} 4 & \xrightarrow{\quad} & 2 & \textcircled{4} \\ 3 & \xrightarrow{\quad} & 2 & \textcircled{3} \\ 2 & \xrightarrow{\quad} & 1 & \textcircled{2} \\ 1 & \xrightarrow{\quad} & 1 & \textcircled{1} \\ 0 & \xrightarrow{\quad} & 0 & \end{matrix}$   
 $\textcircled{4} \textcircled{3} \textcircled{2} \textcircled{1}$   
 $+ \cdot +$

$$(4) \text{ (ordinary)} \oplus (\text{p-rank } 0). \quad (\bar{M}, \langle, \rangle) = (\bar{M}^{\text{ord}}, \langle, \rangle) \oplus (\bar{M}^0, \langle, \rangle) \quad (59)$$

$$f = \text{p-rank}(\bar{M}), \quad 2f = \dim \bar{M}^{\text{ord}}.$$

$$\downarrow$$

$$\dim_h = 2(g-f)$$

$$\varphi(\bar{M}^{\text{ord}}) = (1, \dots, f), \quad \varphi(\bar{M}^0) = (\varphi^0(1), \dots, \varphi^0(g-f))$$

$$\varphi(\bar{M}) = (1, 2, \dots, f, \underbrace{f + \varphi^0(1)}_{\varphi(f+1)}, \dots, \underbrace{\varphi^0(g-f) + f}_{\varphi(g)})$$

$$\text{eg. } g=2, \quad \varphi_1=1, \quad \varphi_2=0 \quad \varphi_1 \oplus \varphi_2 = 11$$

$$f(\varphi) = \text{p-rank}(\varphi) = \max \{ i : \varphi(i) = i \}$$

$$ES^{(f)}(g) := \{ \varphi \in ES(g) : f(\varphi) = f \}$$

$$\# ES(g) = 2^g, \quad \# ES^{(0)}(g) = 2^{g-1}.$$

$$\downarrow$$

$$\{ \varphi : \varphi(1) = 0 \}$$

$$ES^{(0)}(g) = ES(g-1)$$

$$(0, \varphi(1), \dots, \varphi(g-1)) \leftarrow (\varphi(1), \dots, \varphi(g-1))$$

### 7.3 Construction from $\psi$ to $(\bar{M}, \langle, \rangle)$ .

(60)

Ref. [Oort, §9.1], [Hara, 2007, §5.2]

$\psi \in FS$ .

Herbrand fun  
(? role of  $\psi \approx$  organizes  
ramifications/jumps)

Write  $B_0 = \{1 \leq i \leq 2g : \psi(i-1) < \psi(i)\}$  as  $\{m_1 < m_2 < \dots < m_g\}$

$B_1 = \{1 \leq i \leq 2g : \psi(i-1) = \psi(i)\}$  as  $\{n_1 > n_2 > \dots > n_g\}$

Note  $m_i + n_i = 2g+1$ .

Construct polarized DM,  $(A_\psi, \langle, \rangle)$  over  $\mathbb{F}_p$  as follows.

Two bases  $\{Y_g, \dots, Y_1, X_1, \dots, X_g\} = \{Z_1, \dots, Z_{2g}\}$  with following relations.

For  $1 \leq i \leq g$ :

$$(i) \quad Z_{m_i} = X_i, \quad Z_{n_i} = Y_i \quad \forall 1 \leq i \leq g.$$

$$(ii) (a) \quad F X_i = Z_i, \quad F Y_i = 0$$

$$(b) \quad V(Z_i) = 0, \quad V(Z_{2g+1-i}) = \pm Y_i = \varepsilon_i Y_i$$

$$(iii) \quad \begin{aligned} \langle X_i, Y_i \rangle &= 1, & \text{other pairings} &= 0. \\ \text{"} & & & \\ - \langle Y_i, X_i \rangle & & & \end{aligned}$$

$$\varepsilon_i = \langle X_i, \underbrace{\varepsilon_i Y_i}_{V(Z_{2g+1-i})} \rangle = \langle F X_i, Z_{2g+1-i} \rangle = \langle Z_i, Z_{2g+1-i} \rangle$$

The sign  $\varepsilon_i$  is defined by

$$\varepsilon_i = \begin{cases} 1 & \text{if } Z_{2g+1-i} \in \{Y_1, \dots, Y_g\} \\ -1 & \text{if } Z_{2g+1-i} \in \{X_1, \dots, X_g\} \end{cases}$$

Note: the action of  $V$  on  $\{Z_j\}$  is determined by (ii) (a) and (iii), (61)

$$V(Z_i) = V(FX_i) = 0, \quad VZ_{2g+1-i} = \sum a_i X_i + \sum b_i Y_i \quad a_i, b_i \in \mathbb{F}_p.$$

$$-a_i = \langle Y_i, VZ_{2g+1-i} \rangle = \langle FY_i, Z_{2g+1-i} \rangle = 0 \Rightarrow a_1 = a_2 = \dots = a_g = 0$$

$$b_j = \langle X_j, VZ_{2g+1-i} \rangle = \langle Z_j, Z_{2g+1-i} \rangle = 0 \quad \text{if } i \neq j$$

$$\text{For } j=i \text{ if } \{Z_i, Z_{2g+1-i}\} = \{X_j, Y_j\} \text{ then } b_i = 1$$

$$\text{if } \{Z_i, Z_{2g+1-i}\} = \{Y_j, X_j\} \text{ then } b_i = -1$$

$$\text{Rule } Z_{\underset{\substack{\parallel \\ m_i}}{j}} = X_i \quad \text{if } j \in B_0 \quad (\text{break point})$$

$$\Rightarrow FZ_j = FX_i = Z_i$$

$$Z_{\underset{\substack{\parallel \\ n_i}}{j}} = Y_i \quad \text{if } j \notin B_1 \quad (\text{not break point})$$

$$\Rightarrow V^{-1}(Z_j) = V^{-1}(Y_i) = \varepsilon_i Z_{2g+1-i}$$

**Remark** The subspaces  $\bigoplus_{i=1}^j \mathbb{F}_p Z_i$  give the final filtration of  $(A_\varphi, \langle, \rangle)$ .

• Somehow  $\Psi$ : a sort of fen organizing break point.

Compare the Herbrandt function  $\varphi, \psi$  and the

Hasse-Arf Thm. (?)

Example:  $g=3$ ,  $\psi = 0 \ 0 \ 0 \ ① \ ② \ ③$

$B_8 = \{4, 5, 6\}$

$B_1 = \{3, 2, 1\}$

(62)

$z_4 = x_1, z_5 = x_2, z_6 = x_3, z_1 = y_3, z_2 = y_2, z_3 = y_1$

$$\begin{array}{c}
 \begin{array}{cccccc}
 z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 0 & 0 & 0 & -y_{4=3} & -y_2 & -y_1 \\
 \parallel & & & \parallel & \parallel & \parallel \\
 & & & -z_1 & -z_2 & -z_3
 \end{array}
 \end{array}$$

Ignore  $\varepsilon: \langle \cdot, \cdot \rangle$

$$F \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \vee F \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \vee F \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \vee$$

alternative

$$F \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \vee^{-1} F \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \vee^{-1} F \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \vee^{-1}$$

Supersingular  
DM rk 2

$$\begin{cases} e_1 \rightarrow e_2 \\ e_2 \rightarrow -pe_1 \end{cases}$$

$$\begin{cases} e_1 \mapsto e_2 \\ e_2 \rightarrow pe_1 \end{cases}$$

$$\star \quad \varepsilon = \langle e_1, e_2 \rangle, \quad \langle Fe_1, Fe_2 \rangle = p \langle e_1, e_2 \rangle^\sigma$$

$$-p\varepsilon = \langle e_2, pe_1 \rangle = p \cdot \varepsilon^\sigma$$

$$\boxed{\varepsilon^\sigma = -\varepsilon}$$

$$\langle Fe_1, Fe_2 \rangle = p \langle e_1, e_2 \rangle^\sigma = p \varepsilon^\sigma$$

$$p\varepsilon = \langle e_2, -pe_1 \rangle \quad \varepsilon^\sigma = \varepsilon, \text{ can choose } \varepsilon = 1$$

$$\langle e_1, e_2 \rangle = 1$$

## 7.4 Connection with the Weyl group

(63)

$W(Sp_{2g})$  = the Weyl group of  $Sp_{2g} = N_G(T)/T$ ,  $T \subset Sp_{2g}$  max. torus.

$$= \left\{ \pi \in W(GL_{2g}) = S_{2g} : \pi(i) + \pi(2g+1-i) = 2g+1 \right. \\ \left. \cap \left\{ \hat{i}, 2g+1-i \right\} \right\}$$

$$= \langle s_1, s_2, \dots, s_g \rangle$$

$$s_i = (i, i+1)(2g-i, 2g+1-i), \quad (1 \leq i \leq g-1, \quad \begin{matrix} i & 2g+1-i \\ i+1 & 2g-i \end{matrix})$$

$$s_g = (g, g+1)$$

$$s_i^2 = 1 \quad \forall i, \quad (s_i s_j)^{m(i,j)} = 1.$$

It turns out that these are the only relations called a Coxeter gp.

$$\text{where } m(i, j) = \begin{cases} 2 & \text{if } |i-j| > 2 \\ 3 & \text{if } (i, j) = (i, i+1) \neq (g-1, g) \\ 4 & \text{if } (i, j) = (g-1, g) \end{cases}$$

$$C_g: \quad \begin{array}{ccccccc} \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \\ 1 & & 2 & & g-1 & & g \end{array} \quad \text{Dynkin diagram}$$

$$m(i, j) = \begin{array}{c} 2 \\ \circ \quad \circ \\ \cdot \text{---} \circ \\ \cdot \text{---} \circ \\ \cdot \text{---} \circ \end{array}$$

$\circ \rightleftharpoons \circ$  means that "arrow toward the short root", or

" $= + <$  and short root  $<$  long root". Note  $C_n$  has a long root.

$$g=4, \quad s_3 = (34)(56), \quad s_4 = (45)$$

$$s_3 s_4 = (4653), \quad \text{ord} = 4.$$

Observe that  $B_0 = \{ m_1 < m_2 < \dots < m_g \}$  satisfies

(64)

$$B_0 \cap B_0^\vee, \quad B_0 \sqcup B_0^\vee = \{1, 2, \dots, 2g\}$$

$\uparrow$   
 $B_1$

So  $\exists! \pi \in W(Sp_{2g})$  s.t.  $B_0 = \{ \pi^{-1}(1) < \dots < \pi^{-1}(g) \}$

$$B_1 = \{ \pi^{-1}(2g) > \pi^{-1}(2g-1) > \dots > \pi^{-1}(g+1) \}$$

$$\pi^{-1}(2g-1+i) = n_i$$

$$Z_{\pi^{-1}(2g+1-i)} = Z_{n_i} = Y_i$$

$$V^{-1}(Z_{\pi^{-1}(2g+1-i)}) = \pm Z_{2g+1-i} = Z_{\pi(j)}$$

If  $j = \pi^{-1}(2g+1-i)$ , then  $V^{-1}(Z_j) = \pm Z_{\pi(j)}$

**Prop.** There is a bijection between

$$\{ \pi \in W(Sp_{2g}) : \pi^{-1}(1) < \dots < \pi^{-1}(g) \} \text{ with } \mathcal{DM}_1(g).$$

Construction:  $B_0 = \{ \pi^{-1}(i) \mid 1 \leq i \leq g \}$

$$B_1 = \{ \pi^{-1}(2g-i+1) \mid 1 \leq i \leq g \}$$

$(\bar{M}, \langle, \rangle)$ : basis  $\{Z_j\}_{1 \leq j \leq 2g}$ .

$$\langle Z_i, Z_{2g+1-i} \rangle = \varepsilon_i = - \langle Z_{2g+1-i}, Z_i \rangle.$$

$$F(Z_j) = Z_{\pi(j)} \quad \text{if } j \in B_0$$

$$V'(Z_j) = \pm Z_{\pi(j)} \quad \text{if } j \in B_1,$$

$$[Z_j = \pm V Z_{\pi(j)}]$$

$$\pi \leadsto (M_\varphi, \langle, \rangle)$$

What is the NP

$\lambda(\varphi)$  of  $M_\varphi$ ?

Eg. (1)  $B_0 = \{2, 3, 6\} = \{\pi^{-1}(1), \pi^{-1}(2), \pi^{-1}(3)\}$

$$B_1 = \{5, 4, 1\} = \{\pi^{-1}(6), \pi^{-1}(5), \pi^{-1}(4)\}$$

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$$

$$4 \quad 1 \quad 2 \quad 5 \quad 6 \quad 3, \quad \pi = (145632)$$

(2)  $\varphi = 000, \quad \pi_0 = (14)(25)(36)$

Note that  $\ell(\pi_0) \neq 0$   $W(Sp_{2,5})$  as a Coxeter gp

( need some modification so that )  
 $\ell(w) = \dim EO_w$

(3)  $\varphi = 123$ , ordinary,  $\pi = 1$



## 7.5 Geometry of EO strata.

(66)

**Def.** (1) For  $\varphi, \varphi' \in ES$ , define  $\varphi \leq \varphi'$  if  $\varphi(i) \leq \varphi'(i) \quad \forall 1 \leq i \leq g$ .

Define  $\varphi \leq \varphi'$  if  $S_\varphi \cap \overline{S_{\varphi'}} \neq \emptyset$  ( $\varphi$  occurs in the boundary of  $S_{\varphi'}$ ).

(2) Call  $\varphi$  **supersingular** if the EO stratum  $S_\varphi$  is contained in the supersingular locus  $S_g$ .

**Prop** If  $(\bar{M}, \langle, \rangle) : p.p. DM$ , with elementary seq.  $\varphi$ .

(1) The  $a \neq a(\bar{M}) = g - \varphi(g)$ .

(2) The  $p$ -rank  $f(\bar{M}) = \max \{ i : \varphi(i) = i \}$   
 $= \frac{1}{2} \# \text{Fix}(\pi)$ , where  $\pi \in W(Sp_{2g}) \hookrightarrow \varphi$ .

**Thm** For  $\varphi \in ES$ , let  $S_\varphi \subseteq \mathcal{A}_{g,N}$  be the EO stratum ass to  $\varphi$ .

(1)  $S_\varphi$  : non-empty, smooth, quasi-affine of  $\dim |\varphi| = \sum_{i=1}^g \varphi(i)$ .

(2)  $\exists!$  max EO stratum, which is the ordinary locus

$\exists!$  min EO stratum, which is the superspecial locus.

(3)  $\overline{S_\varphi} = \bigsqcup_{\varphi' \leq \varphi} S_{\varphi'}$  (stratification property)

(4) If  $\varphi \leq \varphi'$ , then  $S_\varphi \subseteq \overline{S_{\varphi'}}$ . So  $\varphi \leq \varphi'$ .

The converse is generally not true.  $\varphi \leq \varphi' \not\Rightarrow \varphi \leq \varphi'$

(5)  $\varphi$  is supersingular  $\Leftrightarrow \varphi(i) = 0 \quad \forall \quad i \leq \left\lfloor \frac{g+1}{2} \right\rfloor$  (67)

"first half +  $\varepsilon = 0$ "

$$\left\{ \begin{array}{ll} g=4 & 00** \text{ s.s.} \\ g=5 & 000** \text{ s.s.} \end{array} \right\}$$

(6) If  $\varphi$  is not supersingular, then  $S_\varphi$  is irreducible.

Remark (1) - (4) due to Oort, Ekedahl - Oort

(5) Oort, extended by Harashita ..

(6) Ekedahl - van der Geer (2004)