

§ 2 Introduction to complex Siegel modular varieties, 02.12.2021

(10)

$$2.1 \quad g \geq 1 \text{ integer, } \mathbb{H}_g := \left\{ Z \in \text{Mat}_g(\mathbb{C}) : \begin{array}{l} Z = Z^t \\ \text{Im } Z \gg 0 \end{array} \right\}$$

the Siegel upper half space of degree g .

complex manifold, actually a Hermitian symmetric domain, of $\dim g(g+1)/2$.

When $g=1$, it is the upper half plane.

(V, ψ) : non-deg symplectic space / \mathbb{Q} of $\dim 2g$.

Choose a Lagrangian basis e_1, \dots, e_{2g} s.t.

$$\psi \sim J_g = \left(\begin{array}{c|c} 0 & I_g \\ \hline -I_g & 0 \end{array} \right)$$

$$GS_{P_{2g}}(\mathbb{Q}) = \left\{ r \in \text{Aut}(V_{\mathbb{R}}) : \begin{array}{l} \exists \nu(r) \in \mathbb{Q}^{\times} \text{ s.t.} \\ \psi(rx, ry) = \nu(r) \psi(x, y) \quad \forall x, y \in V_{\mathbb{R}} \\ \psi(x, r^*y) \quad * = \text{symplectic adjoint} \\ r^* \cdot r = \nu(r) \cdot I_g \end{array} \right\}$$

$$S_{P_{2g}}(\mathbb{Q}) = \left\{ r \in GL_{2g}(\mathbb{Q}) : r^* \cdot r = 1_{2g} \right\}$$

$$r = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_{2g}(\mathbb{R}), \quad r^* = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix} \quad \begin{matrix} g=1 \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \end{matrix}$$

$$r \in S_{P_{2g}} \Leftrightarrow \begin{array}{l} A^t \cdot C = C^t \cdot A \quad A^t D - C^t \cdot B = 1_g \\ B^t \cdot D = D^t \cdot B \end{array}$$

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Lemma (1) If $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{R})$, $Z \in \mathbb{H}_g$, then $CZ + D$ is invertible.

(2) The action $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B) \cdot (CZ + D)^{-1}$ is transitive.

and we have $Sp_{2g}(\mathbb{R})/U(g) \simeq \mathbb{H}_g$.

Pf: (1) Suppose not, then \exists column vect v^{*0} st $(CZ + D) \cdot v = 0$

Then $\alpha := v^t \cdot (AZ + B)^t \cdot \overline{(CZ + D)} \cdot \bar{v} \in \mathbb{C}$.

$$0 = \alpha - \alpha^* = v^t \cdot (AZ + B)^t \cdot \overline{(CZ + D)} \cdot \bar{v} - v^t \cdot (CZ + D)^t \cdot \overline{(AZ + B)} \cdot \bar{v}$$

$$= (Z A^t + B^t) (C \bar{Z} + D) - (Z C^t + D^t) (A \bar{Z} + B)$$

$$= \cancel{Z A^t C \bar{Z}} + Z A^t D + B^t C \bar{Z} + \cancel{B^t D}$$

$$- \cancel{Z C^t A \bar{Z}} - Z C^t B - D^t A \bar{Z} - \cancel{D^t B}$$

$$\cancel{Z} - \cancel{\bar{Z}}$$

$$Z = X + iY$$

$$= X + iY - (X - iY) = 2iY$$

$$0 = 2i v^t \cdot Y \bar{v} \quad * \quad \because Y: \text{positive definite}, v \neq 0$$

$$(2) \quad Z = X + iY \in \mathbb{H}_g,$$

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$\therefore Y \gg 0$. Write $Y = u \cdot u^t$ for some $u \in GL_n(\mathbb{R})$.

$$\text{Put } r = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^{t^{-1}} \end{pmatrix} = \begin{bmatrix} u & X \cdot (u^t)^{-1} \\ (u^t)^{-1} \end{bmatrix}.$$

$$r \cdot iI_g = (u i + X (u^t)^{-1}) \cdot (u^t) = X + iY.$$

$$\text{Stab}(iI_g). \quad (Ai + B)(Ci + D)^{-1} = iI_g$$

$$(Ai + B) = iD - C \Leftrightarrow A = D, B = -C$$

$$r = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \Leftrightarrow A + iB \in \text{Mat}_g(\mathbb{C})$$

$$\begin{array}{ccc} \text{Mat}_g(\mathbb{C}) & \hookrightarrow & \text{Mat}_{2g}(\mathbb{R}) \\ A + iB & \mapsto & \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \\ \text{Hermitian} & & \downarrow \text{symplectic} \\ \text{adjoint} \rightarrow *^t & & \text{involution} \\ A^t - iB^t & \mapsto & \begin{bmatrix} A^t & -B^t \\ B^t & A^t \end{bmatrix} \end{array}$$

$$\text{Stab}(iI_g) = \{ r \in GL_g(\mathbb{C})$$

$$\left. \begin{array}{l} r^{*t} \cdot r = I_g \\ \parallel \\ r^* r \\ = U(g) \end{array} \right\}$$

$$\text{Define } r = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GSp}_{2g}(\mathbb{R}) \quad r \cdot Z = (AZ + B) \cdot (CZ + D)^{-1}.$$

$$\text{Then } \text{Im } rZ \gg 0 \Leftrightarrow \omega(r) > 0.$$

$$\text{GSp}_{2g}(\mathbb{R})^+ = \{ r : \omega(r) > 0 \} \subseteq \text{GSp}_{2g}(\mathbb{R}) \text{ the neutral connected component.}$$

$Sp_{2g}(\mathbb{Z}) \curvearrowright \mathbb{H}_g$. The quotient space $Sp_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g$ is
 called the **complex Siegel modular variety** of degree g ^(genus g).

normal analytic space, with finitely many singular points
 of quotient singularities $(\approx \mathbb{C}^N / \text{finite g.p.s.})$ ^{$N \geq 3$} $(\underbrace{Sp_{2g}(\mathbb{Z}) \supset \Gamma(N)}_{\bar{\Gamma}})$
 quasi-projective normal alg var / \mathbb{C} (Baily-Borel) $[\bar{\Gamma} \subset Sp_{2g}(\mathbb{Z}/N\mathbb{Z})]$

When $g=1$, $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ the (open) modular curve.

$SL_2(\mathbb{Z}) \backslash \mathbb{H}$ classifies complex elliptic curves up to isom.

Similarly, $Sp_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g$ classifies g -dim principally
 polarized abelian varieties / \mathbb{C} .

2.2. Description of complex polarized abelian varieties.

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Thm (Riemann) There is an equivalence of categories:

$$\left(\begin{array}{l} \text{complex polarized} \\ \text{abelian varieties} \end{array} \right) \text{ and } \left(\begin{array}{l} (V, \Lambda, E), \text{ where } V: \mathbb{C}\text{-vect. sp.} \\ \Lambda \subset V \text{ full rank lattice,} \\ E: \text{Riemann form on } V/\Lambda \end{array} \right)$$

A : abelian var / \mathbb{C} , $A(\mathbb{C}) = V/\Lambda$ complex torus,

$V = \text{Lie}(A)$ \mathbb{C} -v.s. $\supseteq \Lambda = H_1(A, \mathbb{Z})$ full rank lattice, $\text{Lie}(A) = \Lambda_{\mathbb{R}}$.

Recall a Riemann form on a complex torus V/Λ is a non-degenerate alternating pairing $E: \Lambda \times \Lambda \rightarrow \mathbb{Z}$, st.

(i) $E(ix, iy) = E(x, y) \quad \forall x, y \in \Lambda_{\mathbb{R}}$.

(ii) the symmetric form $E(ix, y)$ is positive definite.

There is a one-one correspondence

$$\Lambda_{\mathbb{R}} = V$$

$$\left\{ \begin{array}{l} \text{(positive definite)} \\ \text{Hermitian forms} \\ \text{on } V \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{alternating form } E: \Lambda_{\mathbb{R}} \times \Lambda_{\mathbb{R}} \rightarrow \mathbb{R} \\ \text{st (i) holds} \\ \text{(and (ii) holds)} \end{array} \right\}$$

$$H \longmapsto \underline{E = \text{Im } H}$$

$$H(x, y) := E(ix, y) + i \underline{E(x, y)} \longleftarrow E$$

\uparrow replace 2nd x by y

$$(H(x, x) = E(ix, x) > 0)$$

Put $V^t = \{ f: V \rightarrow \mathbb{C} \text{ anti-}\mathbb{C}\text{-linear map} \}$

$$\Lambda^t := \{ f \in V^t : (\text{Im } f)(\Lambda) \subseteq \mathbb{Z} \} \simeq \text{Hom}(\Lambda, \mathbb{Z})$$

If $A(\mathbb{C}) = V/\Lambda$, then $A^t(\mathbb{C}) = V^t/\Lambda^t$.

- An isogeny $\lambda: V/\Lambda \rightarrow V^t/\Lambda^t \Leftrightarrow \lambda: V \rightarrow V^t: \mathbb{C}\text{-linear}$
 $\lambda(\Lambda) \subseteq \Lambda^t$.

Put $H(x, y) := (x, \lambda y)$, where $(,) : V \times V^t \rightarrow \mathbb{C}$.

See \mathbb{C} -linear in x , anti- \mathbb{C} -linear in y and

- $\lambda = \lambda^t \Leftrightarrow H: \text{Hermitian}$ (symmetric isogenies \leftrightarrow Hermitian forms)
 $\lambda: \text{polarization} \Leftrightarrow H: \text{positive definite}$.

So $\text{Im } H = \langle , \rangle_\lambda : \Lambda \times \Lambda \rightarrow \mathbb{Z}$ alternating.

$(,) : \Lambda \times \Lambda^t \rightarrow \mathbb{Z}$ canonical pairing.

Then $\langle , \rangle_\lambda = (x, \lambda y)$.

\Rightarrow equivalence of categories:

$$\left(\begin{array}{c} \text{polarized abelian} \\ \text{varieties} / \mathbb{C} \end{array} \right) \longleftrightarrow \left(\begin{array}{c} (V, \Lambda, E) \text{ where} \\ E: \text{Riemann form on } V/\Lambda \end{array} \right)$$

$$(A, \lambda) \longmapsto (\text{Lie } A, H_1(A, \mathbb{Z}), \langle , \rangle_\lambda)$$

$$(V/\Lambda, H) \longleftarrow (V, \Lambda, E)$$

$H = \text{Hermitian form ass to } E$.

$V = \Lambda_{\mathbb{R}} \Rightarrow$ complex structure J on $\Lambda_{\mathbb{R}}$

$$\begin{matrix} \cup & \cup \\ i & J \end{matrix}, \quad J^2 = -I$$

Then $\begin{cases} E(Jx, Jy) = E(x, y) & \forall x, y \in \Lambda_{\mathbb{R}} \\ E(Jx, y) : \text{positive, definite.} \end{cases}$

Such a J on $\Lambda_{\mathbb{R}}$ is called an **admissible complex structure**

on $(\Lambda_{\mathbb{R}}, E)$

\Rightarrow R.H.S of above
equivalence of categories $\longleftrightarrow \left((\Lambda, J, E), \text{ where } J : \right.$
 $\left. \text{admissible complex structure on } (\Lambda_{\mathbb{R}}, E) \right)$

2.3. Modular interpretation of Siegel modular varieties

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$\mathcal{A}_g(\mathbb{C}) =$ the set of isom classes of g -diml PPAVs/ \mathbb{C} .

\downarrow
 (A, λ) Since $(H_1(A, \mathbb{Z}), \langle, \rangle_\lambda)$ are all isomorphic.

Fix (Λ, Ψ) : self-dual symplectic lattice of rank $2g$.

with a standard basis. $Sp(V, \Psi) = Sp_{2g}(\mathbb{Z})$

Consider $\tilde{\mathcal{A}}_g(\mathbb{C}) = \left\{ (A, \lambda, \xi) : \begin{array}{l} (A, \lambda) \in \mathcal{A}_g(\mathbb{C}) \\ \xi : (H_1(A, \mathbb{Z}), \langle, \rangle_\lambda) \simeq (\Lambda, \Psi) \end{array} \right\}$

Then $Sp_{2g}(\mathbb{Z}) \curvearrowright \tilde{\mathcal{A}}_g(\mathbb{C})$ on the left, and

$$Sp_{2g}(\mathbb{Z}) \backslash \tilde{\mathcal{A}}_g(\mathbb{C}) \simeq \mathcal{A}_g(\mathbb{C})$$

$\mathcal{D} := \{ \text{admissible complex structures on } (\Lambda_{\mathbb{R}}, \Psi) \}$

Then $\tilde{\mathcal{A}}_g(\mathbb{C}) \cong \mathcal{D}$

$(A, \lambda, \xi) \longmapsto J$ complex str on $\Lambda_{\mathbb{R}}$ via

$$\begin{array}{ccc} \xi_{\mathbb{R}} : & \text{Lie}(A) & \xrightarrow{\sim} \Lambda_{\mathbb{R}} \\ & \parallel & \nearrow \xi_{\mathbb{R}} \\ & H_1(A, \mathbb{R}) & \end{array}$$

$(\Lambda_{\mathbb{R}}/\Lambda, H_\Psi, \text{id})$

$\nwarrow \quad \nearrow J$
 $(\Lambda_{\mathbb{R}}, J, \Psi, \text{id})$

$$\Rightarrow S_{p_{2g}}(\mathbb{Z}) \backslash H_g \simeq S_{p_{2g}}(\mathbb{Z}) \backslash S_{p_{2g}}(\mathbb{R}) / U(g) \simeq S_{p_{2g}}(\mathbb{Z}) \backslash \mathfrak{D}$$

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$$\simeq S_{p_{2g}}(\mathbb{Z}) \backslash \tilde{A}_g(\mathbb{C}) \simeq A_g(\mathbb{C}). \quad \square$$

Def: G : connected reductive alg gp / \mathbb{R} . An automorphism

$\Theta: G \rightarrow G$ of order 2 is called a **Cartan involution** if

$K := \{g \in G(\mathbb{R}) : \Theta(g) = g\}$ is a maximal compact subgp.

Lemma Θ, Θ' two Cartan involutions on G . Then $\exists g \in G(\mathbb{R})$

$$\text{s.t. } \Theta' = \text{Int}(g) \circ \Theta \circ \text{Int}(g)^{-1}.$$

$$\left(\text{Int}(g) \circ \Theta \circ \text{Int}(g)(x) = g \Theta(g^{-1} x g) g^{-1} = g \Theta(g)^{-1} \cdot \Theta(x) \Theta(g) g^{-1} \right)$$

Pf of the fact: transitivity of $S_{p_{2g}}(\mathbb{R}) \curvearrowright \mathfrak{D}$:

Suppose J another admissible complex structure on $(V_{\mathbb{R}}, \Psi)$

$$J \in S_{p_{2g}}(\mathbb{R}), \quad J^2 = -I, \quad \Psi(Jx, y) \text{ positive definite.}$$

$$\Rightarrow \text{Int}(J): \text{Cartan involution.}$$

By Lemma, $\exists \gamma \in S_{p_{2g}}(\mathbb{R})$ s.t. $\text{Int}(J) = \text{Int}(\gamma) \circ \text{Int}(J_0) \circ \text{Int}(\gamma)^{-1}$

$$\Rightarrow J = \gamma J_0 \gamma^{-1} \cdot c, \quad c \in Z(S_{p_{2g}}(\mathbb{R})) = \{\pm 1\}.$$

If $c = -1$, then $\Psi(Jx, y)$: negative definite. ~~*~~

$$\Rightarrow c = 1, \quad J = \gamma J_0 \gamma^{-1}.$$

An elementary attempt for the proof:

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$$J_0 = J_g = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix} \quad \text{Write } J = J_0 B \Rightarrow \begin{cases} B \in Sp_{2g}(\mathbb{R}) \\ J_0^t J_0 = I \\ J_0^t = -J_0 \end{cases}$$

$\Psi(Jx, \overset{\text{column vector}}{y}) = x^t J^t J_0 y = x^t B^t J_0^t J_0 y = x^t B^t y$ $B = B^t$ positive definite.

Find $\gamma \in Sp_{2g}(\mathbb{R})$ s.t. $J = \gamma J_0 \gamma^{-1}$.

$$\gamma^t J_0 \gamma = J_0 \Rightarrow \gamma^{-1} = J_0^{-1} \gamma^t J_0 = \gamma^* \quad J_0^{-1} = J_0^t = -J_0$$

$$J = \gamma J_0 \gamma^{-1} = \gamma J_0 J_0^{-1} \gamma^t J_0 = J_0 J_0^{-1} \gamma \gamma^t J_0 = J_0 (J_0^t \gamma) (J_0^t \gamma)^t$$

\Rightarrow Solvability of $J = \gamma J_0 \gamma^{-1}$ for some $\gamma \in Sp_{2g}(\mathbb{R})$

$(\Leftrightarrow) B = u \cdot u^t$ for some $u \in Sp_{2g}(\mathbb{R})$. (putting $\gamma = J_0 u$)

The problem is equiv.: If $B \in Sp_{2g}$, $B = B^t \gg 0$, $J_0 B J_0 B = -I$

then $\exists u \in Sp_{2g}$ s.t. $B = u \cdot u^t$.

Note: $J_0 u \underbrace{u^t J_0 u}_{= -I} u^t = J_0 u J_0 u^t = J_0^2 = -I$.

$\Rightarrow J_0 B J_0 B = -I$: nec. condition.

$\bullet \{ u \in GL_{2g}(\mathbb{R}) : B = u \cdot u^t \} =$ non-empty $O(2g)$ -torsor

$\{ u \in Sp_{2g}(\mathbb{R}) : B = u \cdot u^t \}$ if non-empty, is $U(g)$ -torsor.

Exercise: Let $B \in Sp_{2g}$, $B = B^t \gg 0$, $J_0 B J_0 B = -I$. Show that

there exists $u \in Sp_{2g}$ s.t. $B = u \cdot u^t$.

2.4 Variants.

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$$N \geq 1, \quad \Gamma(N) = \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z}) : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

Choose a primitive N^{th} root of unity $\zeta_N \in \overline{\mathbb{Q}} \subseteq \mathbb{C}$.

A full level- N structure on (A, λ) is an isom
(w.r.t. ζ_N)

$$h: \Lambda / N\Lambda \xrightarrow{\sim} A[N](\mathbb{C}) \quad \text{s.t.}$$

$$(\Lambda / N\Lambda)^2 \longrightarrow A[N](\mathbb{C})^2$$

$$\downarrow \varphi$$

$$\downarrow e_\lambda = \langle \cdot, \lambda \rangle \cdot "2\pi i"$$

$$\mathbb{Z} / N\mathbb{Z} \xrightarrow[\sim]{\zeta_N} M_N(\mathbb{C})$$

$\mathcal{A}_{g,N}(\mathbb{C}) = \text{moduli space of } g\text{-diml PPAVs} / \mathbb{C} (A, \lambda, h)$
with a full level- N structure.

$$\cong \Gamma(N) \backslash \mathbb{H}_g.$$

For $N \geq 3$, $\Gamma(N)$ has no torsion elements (except 1)

the $\Gamma(N)$ -action on \mathbb{H}_g is free, and $\mathcal{A}_{g,N}(\mathbb{C})$: smooth

quasi-proj. alg variety / \mathbb{C} .

§3 Moduli spaces of abelian varieties.

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3.1 Moduli spaces.

S : base scheme, $F: (S\text{-sch}) \rightarrow (\text{Set})$ contravariant functor.

$$\begin{array}{c} T \\ \downarrow \\ S \end{array} \mapsto F(T) = \left\{ \begin{array}{l} \text{(isom classes) of a family of} \\ \text{geometric objects of certain kind} \\ \text{which are parameterized by } T \end{array} \right\}$$

or $(\mathbb{C}\text{-Sch})$

e.g. $F: (\mathbb{C}\text{-Sch}) \rightarrow (\text{Set})$

$$F(S) = \left\{ \begin{array}{l} \text{isom classes of} \\ \text{"elliptic curves" over } S \end{array} \right\}, \text{ or}$$

$$F(S) = \left\{ \begin{array}{l} \text{isom classes of smooth proper} \\ \text{algebraic curves over } S \text{ of genus } g \geq 1 \end{array} \right\}$$

Such a functor of classifying geometric objects is called a **moduli problem**, or a **moduli functor**.

Def: Let $F: (S\text{-sch}) \rightarrow (\text{Set})$ be a moduli functor.

(1) If F is representable by an S -scheme $\overset{M}{\underset{\wedge}{\text{}}} \left(\text{i.e. } \exists : F \simeq h_M(T) = \text{Hom}(T, M) \right)$

then M is called the **fine moduli space (scheme)** for F .

The object $\mathcal{X} \rightarrow M$ corresponding to $\text{id}_M \in h_M(M) \simeq F(M)$ is called the **universal family**.

(2) A coarse moduli space (scheme) for F is an S -scheme M (23)

together with a natural transformation $\Theta: F \rightarrow h_M$, satisfying

(a) (universal property) For any S -scheme M' with a natural transformation $\Theta': F \rightarrow h_{M'}$, there exists a unique

morphism $h: M \rightarrow M' / S$ s.t. $\Theta' = h(f) \circ \Theta$.

$$\begin{array}{ccc} F & \xrightarrow{\Theta} & h_M \\ \Theta' \searrow & & \swarrow h(f) \\ & h_{M'} & \leftarrow f \\ & & M' \end{array}$$

(b) $\forall: \text{Spec } k \rightarrow S$ with k alg closed field, the map

$$\Theta(\text{Spec } k): F(\text{Spec } k) \xrightarrow{\sim} h_M(\text{Spec } k) = \text{Hom}(\text{Spec } k, M)$$

is bijective.

Remark: Following from the definition, a fine moduli space or a coarse moduli space, if exists, is unique up to a unique isom.

For modular curves, $Y(N)$ with any $N \geq 3$, is the

fine moduli space, and $Y(1)$ is the coarse moduli space, but

not a fine moduli space. In fact $Y(1) = SL_2(\mathbb{Z}/N\mathbb{Z}) \backslash Y(N)$,

but the universal family $\mathcal{E} \rightarrow Y(N)$ does not descend to an

elliptic curve $\mathcal{E}' \rightarrow Y(1)$.

3.2 Families of abelian varieties.

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Def. (1) An **abelian scheme** over S is a group scheme $\pi: A \rightarrow S$ of finite presentation which is proper and smooth with (geometrically) connected fibers.

We usually assume that S is locally Noetherian, so that

it is equivalent that π is of finite type.

(2) For an abelian scheme A/S , we define the relative

Picard functor in the same way. Define $\text{Pic}^0(A/S)(T)$

$$= \left\{ (\mathcal{L}, \xi) \in \text{Pic}(A/S)(T) \mid \forall t \in T, \mathcal{L}_t \text{ is alg equiv. to trivial one on } A_{t, t} \right\}$$

The functor $\text{Pic}^0(A/S)$ is representable by an algebraic space A^t .

If A^t is a scheme, then A^t is an abelian scheme, called

dual abelian scheme of A/S .

(3) A **polarization** on an abelian scheme A/S is an isogeny
i.e.

(finite & faithfully flat) $\lambda: A \rightarrow A^t$ from A to its dual abelian

scheme s.t. $\forall \bar{s} \in S$ geom point. $\lambda_{\bar{s}} = \lambda_{\mathcal{L}_{\bar{s}}}$ for some ample

invertible sheaf $\mathcal{L}_{\bar{s}}$ on $A_{\bar{s}}$.

Remark: Whether A^t is a scheme or not is a very technical question. It turns out that the answer is always "YES".

An idea is to use the technique of induction in EGA IV.

to reduce to the case where S is Noetherian.

For $g \geq 1$, $N \geq 1$, $\zeta_N \in \bar{\mathbb{Q}} \subset \mathbb{C}$ primitive N^{th} root of unity.

Consider the moduli functor $F_{g,N}: (\mathbb{Z}[\zeta_N, \frac{1}{N}] - \text{Sch}) \rightarrow (\text{Set})$

$$F_{g,N}(S) = \left\{ \begin{array}{l} \text{isom classes of p.p. abel.} \\ \text{scheme } (A, \lambda, \eta) \text{ of rel dim } g \text{ with} \\ \text{a full level-} N \text{ structure over } S \end{array} \right\} \begin{array}{l} \text{level-} N \text{ str on } (A, \lambda) \\ \text{over conn. } S. \\ \eta: \Lambda_{A/N} \xrightarrow{\sim} A[N](S) \\ \text{pulling back } e_x \text{ is} \\ \psi \text{ by } \zeta_N \end{array}$$

Thm (Mumford, Faltings and Chai) (1) If $N \geq 3$, the moduli

functor $F_{g,N}$ is representable by a quasi-projective smooth

scheme $\pi: \mathcal{A}_{g,N} \rightarrow \text{Spec } \mathbb{Z}[\zeta_N, \frac{1}{N}]$ of relative dim $g(g+1)/2$.

With irreducible geometric fibers.

(2) For $N=1, 2$, $F_{g,N}$ admits the coarse moduli scheme

$\mathcal{A}_{g,N} \rightarrow \text{Spec } \mathbb{Z}[\zeta_N, \frac{1}{N}]$ which is quasi-projective of relative
dim $g(g+1)/2$ with irreducible geometric fibers

Remark. The irreducibility of fibers $\mathcal{A}_{g,N} \otimes_{\mathbb{F}_p} \bar{\mathbb{F}_p}$ proved by Faltings - Chai
using the arithmetic compactification.