

Theory on Well-posedness of Boussinesq equations

3.5 Anisotropic Dissipation Case

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Horizontal dissipation

Consider the IVP for the 2D Boussinesq equations with horizontal dissipation

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p + \nu u_{xx} + \theta \vec{e}_2, \\ \nabla \cdot u = 0, \\ \partial_t \theta + (u \cdot \nabla) \theta = 0, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x) \end{cases} \quad (1.1)$$

(1.1) has been shown to possess a unique global solution for suitable (u_0, θ_0) and the following theorem combines the following results:

- Danchin and Paicu, M^3AS (2011).
- Larinos, Lunasin and Titi, JDE (2013).

Theorem 1 (R. Danchin and M. Paicu; Larinos, Lunasin and Titi)

Let $u_0 \in H^1(\mathbb{R}^2)$ and $\nabla \cdot u_0 = 0$. Assume $\omega_0 = \nabla \times u_0 \in \sqrt{L}$, namely

$$\sup_{q \geq 2} \frac{\|\omega_0\|_{L^q}}{\sqrt{q}} < +\infty.$$

Let $\theta_0 \in L^2 \cap L^\infty$. Then the IVP (1.1) has a unique solution (u, θ) satisfying

$$u \in L_{loc}^\infty([0, \infty); H^1), \quad \omega \in L_{loc}^\infty([0, \infty); \sqrt{L}), \quad u_2 \in L_{loc}^2([0, \infty); H^2),$$

$$\theta \in C_b([0, \infty); L^2), \quad \theta \in L^\infty([0, \infty); L^\infty).$$

Remark: The paper of [R. Danchin and M. Paicu](#) (2011) originally assumed that $\theta_0 \in H^s$ with $s \in (1/2, 1)$ to show the uniqueness. Later [Larinos, Lunasin and Titi](#) (2013) was able to prove the uniqueness without this assumption.

First, we prove the global existence of weak solutions in a very weak functional setting via Friedrichs' Method. This method cuts off the high frequencies and thus smooths the functions. The global existence result can be stated as follows.

Theorem 2 (Global weak solution)

Let $\theta_0 \in L^2 \cap L^\infty$ and $u_0 \in H^1$ and $\nabla \cdot u_0 = 0$. Then (1.1) has a global weak solution (u, θ) satisfying

$$\theta \in L^\infty([0, \infty); L^2 \cap L^\infty),$$

$$u \in L^\infty_{loc}([0, \infty); H^1), \quad u_2 \in L^2_{loc}([0, \infty); H^2).$$

Proof. (Friedrichs' Method) Let $n \in \mathbb{N}$ and define

$$L_n^2 = \{f \in L^2(\mathbb{R}^2) \mid \text{supp} \hat{f} \subset B(0, n)\},$$

$$J_n f = (\chi_{B(0, n)} \hat{f})^\vee,$$

where \hat{f} and f^\vee denotes the Fourier and the inverse Fourier transforms, respectively, and $\chi_{B(0, n)}$ is the characteristic function on $B(0, n)$. Clearly, $J_n f \in H^\infty = \bigcap_{s \geq 0} H^s$.

Consider the equations

$$\begin{cases} \partial_t \theta + J_n \nabla \cdot (J_n u J_n \theta) = 0, \\ \partial_t u + \mathcal{P} J_n \nabla \cdot (J_n \mathcal{P} u \otimes J_n \mathcal{P} u) = \nu J_n \mathcal{P} \partial_{xx} u + J_n \mathcal{P} (\theta \vec{e}_2^*), \\ u(x, 0) = J_n u_0, \quad \theta(x, 0) = J_n \theta_0, \end{cases} \quad (1.2)$$

where \mathcal{P} denotes the Leray projection. By the Picard theorem, there exists $T^* > 0$ and a solution $(u, \theta) \in C^1([0, T], L_n^2)$

Noticing that $J_n f = f$ if $f \in L_n^2$ and $\mathcal{P}F = F$ if $\nabla \cdot F = 0$, we have

$$\begin{cases} \partial_t \theta + J_n \nabla \cdot (u \theta) = 0, \\ \partial_t u + \mathcal{P} J_n \nabla \cdot (u \otimes u) = \nu \partial_{xx} u + \mathcal{P}(\theta \vec{e}_2). \end{cases}$$

By the energy method

$$\|\theta\|_{L^2} \leq \|J_n \theta_0\|_2 \leq \|\theta_0\|_{L^2},$$

$$\|u\|_{L^2}^2 + 2\nu \int_0^t \|\partial_x u\|_{L^2}^2 dt \leq (\|u_0\|_{L^2} + t\|\theta_0\|_{L^2})^2.$$

Taking the curl of the u -equation yields

$$\partial_t \omega + \mathcal{P} J_n(u \cdot \nabla \omega) = \nu \partial_{xx} \omega + \partial_x \theta$$

and thus

$$\begin{aligned} \|\omega\|_2^2 + 2\nu \int_0^t \|\partial_x \omega\|_2^2 d\tau &\leq \int_0^t \|\theta_0\|_2 \|\partial_x \omega\|_2 d\tau \\ &\leq \nu \int_0^t \|\partial_x \omega\|_2^2 d\tau + \frac{C}{\nu} \int_0^t \|\theta_0\|_2^2 d\tau. \end{aligned}$$

Therefore, $\theta \in L^\infty([0, \infty); L^2)$ and $u \in L^\infty([0, T]; H^1)$ for any $T > 0$. By the Picard Extension Theorem, (θ, u) is global in time and admits bounds that are uniform in n ,

$$\theta^{(n)} \in L^\infty([0, \infty); L^2), \quad u^{(n)} \in L^\infty([0, T]; H^1).$$

In addition, it can be shown that

$$\partial_t \theta^{(n)} \in L^\infty([0, T]; H^{-3/2}), \quad \partial_t u^{(n)} \in L^\infty([0, T]; H^{-1})$$

Since $L^2 \hookrightarrow H^{-3/2}$ locally and $H^1 \hookrightarrow H^{-1}$ locally, the Aubin-Lions compactness lemma then implies $u^{(n)} \rightarrow u$ in H_{loc}^L for any $-1 \leq L < 1$ and $\theta^{(n)} \rightarrow \theta$ in H_{loc}^L for any $-3/2 \leq L < 0$. We can use these convergence to pass the limit in the weak formulation. This completes the proof. \square

Here, we show that (1.1) has a unique local classical solution.

Theorem 1.1 (Unique local classical solution)

Let $(u_0, \theta_0) \in H^s \times H^{s-1}$ with $s > 2$. Then there is $T > 0$ and a unique solution $(u, \theta) \in C([0, T], H^s \times H^{s-1})$ satisfying (1.1).

Proof. We can either use the mollifier approach as in the book of Majda and Bertozzi or Friedriches' approach above. The crucial part is a local priori bound. Define

$$J^s f = (1 - \Delta)^{s/2} f \quad \text{or} \quad \widehat{J^s f}(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi).$$

Then clearly $\|J^s f\|_{L^2} = \|f\|_{H^s}$.

It follows from the vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega = \nu \partial_{x_1 x_1} \omega + \partial_{x_1} \theta$$

that

$$\frac{1}{2} \frac{d}{dt} \|J^{s-1} \omega\|_2^2 + \nu \|\partial_{x_1} J^{s-1} \omega\|_2^2 = K_1 + K_2,$$

where

$$K_1 = \int \partial_{x_1} J^{s-1} \theta J^{s-1} \omega,$$

$$K_2 = - \int (J^{s-1} (u \cdot \nabla \omega) - u \cdot \nabla J^{s-1} \omega) J^{s-1} \omega.$$

K_1 and K_2 can be bounded as follows. By integration by parts,

$$|K_1| \leq \frac{\nu}{2} \|\partial_{x_1} J^{s-1} \omega\|_2^2 + \frac{1}{2\nu} \|J^{s-1} \theta\|_2^2.$$

Applying the commutator estimates yields

$$\begin{aligned} |K_2| &\leq \|J^{s-1}(u \cdot \nabla \omega) - u \cdot \nabla J^{s-1} \omega\|_2 \|J^{s-1} \omega\|_2 \\ &\leq C(\|J^s u\|_2 \|\omega\|_\infty + \|J^{s-1} \omega\|_2 \|\nabla u\|_\infty) \|J^{s-1} \omega\|_2 \\ &\leq C \|\nabla u\|_\infty \|J^{s-1} \omega\|_2^2. \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \|J^{s-1} \theta\|_2^2 \\
 = & - \int (J^{s-1}(u \cdot \nabla \theta) - u \cdot \nabla J^{s-1} \theta) J^{s-1} \theta \\
 \leq & C \|J^{s-1} \theta\|_2 (\|J^s u\|_2 \|\theta\|_\infty + \|J^{s-1} \theta\|_2 \|\nabla u\|_\infty) \\
 \leq & C \|\theta\|_\infty (\|J^{s-1} \omega\|_2^2 + \|J^{s-1} \theta\|_2^2) + C \|\nabla u\|_\infty \|J^{s-1} \theta\|_2^2
 \end{aligned}$$

Therefore, $Y(t) = \|J^{s-1} \omega\|_2^2 + \|J^{s-1} \theta\|_2^2$ satisfies

$$\frac{d}{dt} Y(t) \leq C (1 + \|\theta\|_\infty + \|\nabla u\|_\infty) Y(t).$$

Since $s > 2$,

$$\|\nabla u\|_{\infty} \leq C \|\nabla u\|_{H^{s-1}} \leq C \|J^{s-1} \omega\|_2.$$

and consequently

$$\frac{d}{dt} Y(t) \leq C \left(1 + \|\theta\|_{\infty} + \sqrt{Y(t)} \right) Y(t).$$

This inequality implies that $\exists T^*$ such that $Y(t) \leq C$ for $t < T^*$.

This completes the proof. \square

Here we establish a global bound for $\|u\|_{H^s}$ and $\|\theta\|_{H^{s-1}}$, which allows us to extend the local solution in the previous subsection to a global one.

Theorem 1.2 (Global classical solution)

Assume that $(u_0, \theta_0) \in H^s \times H^{s-1}$ with $s > 2$. Then (1.1) has a unique global solution $(u, \theta) \in C([0, \infty), H^s \times H^{s-1})$.

Proof. It suffices to obtain a global bound for (u, θ) in $H^s \times H^{s-1}$. From the previous proof, $Y(t) = \|J^{s-1}\omega\|_2^2 + \|J^{s-1}\theta\|_2^2$ satisfies

$$\frac{d}{dt} Y(t) \leq C(1 + \|\theta\|_\infty + \|\nabla u\|_\infty) Y(t). \quad (1.3)$$

The trick is still to control $\|\nabla u\|_\infty$ through the following logarithmic interpolation inequality, for $\sigma > 1$,

$$\|f\|_\infty \leq \sup_{q \geq 2} \frac{\|f\|_{L^q}}{\sqrt{q}} \sqrt{\log(1 + \|f\|_{H^\sigma})}.$$

which proof gives later.

In particular, for $s > 2$, we have

$$\begin{aligned}\|\nabla u\|_{\infty} &\leq C \sup_{q \geq 2} \frac{\|\nabla u\|_q}{\sqrt{q}} \sqrt{\log(1 + \|\nabla u\|_{H^{s-1}})} \\ &\leq C \sup_{q \geq 2} \frac{\|\nabla u\|_q}{\sqrt{q}} \sqrt{\log(1 + \|\omega\|_{H^{s-1}})}.\end{aligned}\quad (1.4)$$

The goal is then to show that, for any $T > 0$,

$$\int_0^T \sup_{q \geq 2} \frac{\|\nabla u\|_q}{\sqrt{q}} dt < \infty. \quad (1.5)$$

This is accomplished through two steps. First, we have

$$\|\omega(T)\|_2^2 + 2\nu \int_0^T \|\partial_{x_1} \omega\|_2^2 dt \leq \frac{2T}{\nu} \|\theta_0\|_2^2$$

and, from the L^q estimate of ω , $2 \leq q < \infty$,

$$\|\omega\|_q^2 \leq \|\omega_0\|_q^2 + \frac{2(q-1)}{\nu} t \|\theta_0\|_q^2$$

$$\|\omega\|_q \leq \|\omega_0\|_q + \sqrt{\frac{2}{\nu} t (q-1)} \|\theta_0\|_q.$$

Thus, we have

$$\sup_{q \geq 2} \frac{\|\omega\|_q}{\sqrt{q}} \leq \sup_{q \geq 2} \frac{\|\omega_0\|_q}{\sqrt{q}} + \sqrt{\frac{2t}{\nu}} \|\theta_0\|_{L^2 \cap L^\infty}.$$

We caution that the inequality $\|\nabla u\|_q \leq \frac{c q^2}{q-1} \|\omega\|_q$ does not really help. Instead, we use the imbedding inequality and prove in the following Lemma, for a constant C independent of q ,

$$\sup_{q \geq 2} \frac{\|f\|_q}{\sqrt{q}} \leq C \|f\|_{H^1}$$

and the simple fact that $\|\partial_{x_1} \omega\|_2 = \|\nabla \partial_{x_1} u\|_2$, we have

$$\begin{aligned} \int_0^T \sup_{q \geq 2} \frac{\|\partial_{x_1} u\|_q}{\sqrt{q}} dt &\leq C \int_0^T \|\partial_{x_1} \nabla u\|_2 dt \\ &= C \int_0^T \|\partial_{x_1} \omega\|_2 dt < \infty. \end{aligned}$$

Due to the divergence-free condition $\partial_{x_1} u_1 + \partial_{x_2} u_2 = 0$, we also have

$$\int_0^T \sup_{q \geq 2} \frac{\|\partial_{x_2} u_2\|_q}{\sqrt{q}} dt < \infty.$$

In addition, $\partial_{x_2} u_1 = \partial_{x_1} u_2 - \omega$ and thus

$$\int_0^T \sup_{q \geq 2} \frac{\|\partial_{x_2} u_1\|_q}{\sqrt{q}} dt \leq \int_0^T \sup_{q \geq 2} \frac{\|\partial_{x_1} u_2\|_q}{\sqrt{q}} dt + \int_0^T \sup_{q \geq 2} \frac{\|\omega\|_q}{\sqrt{q}} dt < \infty.$$

Thus we have proven (1.5). Combining (1.3), (1.4) and (1.5) yields the desired global bound. This completes the proof of Theorem 1.2. \square

Lemma 1.3

For $\sigma > 1$,

$$\|f\|_{\infty} \leq C(\sigma) \sup_{q \geq 2} \frac{\|f\|_{L^q}}{\sqrt{q}} \sqrt{\log(1 + \|f\|_{H^{\sigma}})}, \quad (1.6)$$

$$\sup_{q \geq 2} \frac{\|f\|_{L^q}}{\sqrt{q}} \leq C \|f\|_{H^1}, \quad (1.7)$$

where C 's are constants.

Proof. For proving (1.6), one may split f into low and high frequencies according to the Littlewood-Paley decomposition. More precisely, for any $q \in \mathbb{N}$ one may write

$$f = S_q f + \sum_{p \geq q} \Delta_p f.$$

We thus have

$$\|f\|_{L^\infty} \leq \|S_q f\|_{L^\infty} + \sum_{p \geq q} \|\Delta_p f\|_{L^\infty},$$

whence, using the Bernstein inequalities,

$$\|f\|_{L^\infty} \leq C \frac{\|f\|_{L^q}}{\sqrt{q}} \sqrt{q} + C \sum_{p \geq q} 2^p \|\Delta_p f\|_{L^2}.$$

For simplicity, we set $\|f\|_{\sqrt{L}} := \sup_{q \geq 2} \frac{\|f\|_{L^q}}{\sqrt{q}}$. By $\sigma > 1$, we get

$$\|f\|_{L^\infty} \leq C\|f\|_{\sqrt{L}}\sqrt{q} + C2^{q(1-\sigma)}\|f\|_{H^\sigma}.$$

Now, if $C\|f\|_{H^\sigma} \leq \|f\|_{\sqrt{L}}\sqrt{\log(1 + \|f\|_{H^\sigma})}$, then take q the integer part of $\log(1 + \|f\|_{H^\sigma})$. Otherwise, one may choose for q the integer part of

$$\frac{1}{\sigma - 1} \log_2 \left(\frac{C\|f\|_{H^\sigma}}{\|f\|_{\sqrt{L}}\sqrt{\log(1 + \|f\|_{H^\sigma})}} \right),$$

and we get the desired result (1.6).

For any $q \in [2, \infty)$ and $f \in H^1$, using the Littlewood-Paley decomposition and a Bernstein inequality enables us to write

$$\begin{aligned}\|f\|_{L^q} &\leq \sum_{p \geq -1} \|\Delta_p v\|_{L^q} \\ &\leq C \sum_{p \geq -1} 2^{-2p/q} 2^p \|\Delta_p v\|_{L^2} \\ &\leq C \left(\sum_{p \geq -1} 2^{-4p/q} \right)^{1/2} \|f\|_{H^1} \\ &\leq C \sqrt{q-1} \|f\|_{H^1},\end{aligned}$$

which completes the proof of inequality (1.7). \square

Here we outline the proof of Theorem 1. It consists of two main steps. The first step proves the existence, the second proves the uniqueness.

To prove the existence, we first regularize the initial condition. For $\epsilon > 0$, set

$$J_\epsilon f = \rho_\epsilon * f$$

where $\rho_\epsilon = \frac{1}{\epsilon^2} \rho_0(\frac{x}{\epsilon})$ with $\rho_0 \in C_0^\infty(\mathbb{R}^2)$, $\int_{\mathbb{R}^2} \rho_0(x) dx = 1$ and

$$\rho(x) = \begin{cases} 1, & \text{if } |x| \leq \frac{1}{2}, \\ 0, & \text{if } |x| > 1. \end{cases}$$

Consider (1.1) with the initial data $u^\epsilon(x, 0) = J_\epsilon * u_0$ and $\theta^\epsilon(x, 0) = J_\epsilon * \theta_0$. Since, for any $s > 0$,

$$\|u_0^\epsilon\|_{H^s} \leq C \frac{1}{\epsilon^{s-1}} \|u_0\|_{H^1}, \quad \|\theta_0^\epsilon\|_{H^s} \leq C \frac{1}{\epsilon^s} \|\theta_0\|_{L^2},$$

where C is a constant independent of ϵ . By Theorem 1.2, there exists a unique solution

$$(u^\epsilon, \theta^\epsilon) \in C([0, \infty); H^s \times H^{s-1}).$$

Since $(u^\epsilon, \theta^\epsilon)$ admits a global uniform bound in $H^1 \times L^2$, $(u^\epsilon, \theta^\epsilon) \rightharpoonup (u, \theta)$ in $H^1 \times L^2$ and (u, θ) is a weak solution according to Theorem 2.

Uniqueness.

Danchin and Paicu (2011) assumed that

$$\theta_0 \in H^s \cap L^\infty \quad \text{with} \quad s \in \left(\frac{1}{2}, 1\right).$$

Larios, Lunasin and Titi (2013) assumed that

$$\theta_0 \in L^2 \cap L^\infty.$$

The essential idea is the Yudovich approach. Let $(u^{(1)}, \theta^{(1)})$ and $(u^{(2)}, \theta^{(2)})$ be two solutions. Then the difference $(\tilde{u}, \tilde{\theta})$,

$$\tilde{u} = u^{(1)} - u^{(2)}, \quad \tilde{\theta} = \theta^{(1)} - \theta^{(2)},$$

satisfies

$$\begin{cases} \partial_t \tilde{u} + u^{(1)} \cdot \nabla \tilde{u} + \tilde{u} \cdot \nabla u^{(2)} = \partial_{xx} \tilde{u} - \nabla \tilde{p} + \tilde{\theta} \vec{e}_2, \\ \partial_t \tilde{\theta} + u^{(1)} \cdot \nabla \tilde{\theta} + \tilde{u} \cdot \nabla \theta^{(2)} = 0. \end{cases}$$

The second equation demands too much regularity on θ . To exchange into a variable with less regularity requirement, we introduce

$$\Delta \xi^{(1)} = \theta^{(1)}, \quad \Delta \xi^{(2)} = \theta^{(2)}, \quad \tilde{\xi} = \xi^{(1)} - \xi^{(2)}.$$

Clearly, $\tilde{\xi}$ satisfies

$$\partial_t \Delta \tilde{\xi} + u^{(1)} \cdot \nabla (\Delta \tilde{\xi}) + \tilde{u} \cdot \nabla \Delta \xi^{(2)} = 0.$$

The goal is to show $\tilde{u} \equiv 0$ and $\tilde{\xi} \equiv 0$.

It follows from simple energy estimates that

$$\frac{1}{2} \frac{d}{dt} \|\tilde{u}\|_2^2 + \|\partial_x \tilde{u}\|_2^2 = - \int \tilde{u} \cdot \nabla u^{(2)} \cdot \tilde{u} + \int \Delta \tilde{\xi} \tilde{e}_2 \cdot \tilde{u},$$

$$\frac{1}{2} \frac{d}{dt} \|\nabla \tilde{\xi}\|_2^2 = - \int u^{(1)} \cdot \nabla \Delta \tilde{\xi} \tilde{\xi} - \int \tilde{u} \cdot \nabla \Delta \xi^{(2)} \tilde{\xi}.$$

We now estimates the terms on the right.

$$\begin{aligned} \int \Delta \tilde{\xi} \tilde{e}_2 \cdot \tilde{u} &= \int \Delta \tilde{\xi} \tilde{u}_2 = \int (\partial_{x_1 x_1} + \partial_{x_2 x_2}) \tilde{\xi} \tilde{u}_2 \\ &= - \int \partial_{x_1} \tilde{\xi} \partial_{x_1} \tilde{u}_2 + \int \partial_{x_2} \tilde{\xi} \partial_{x_1} \tilde{u}_1 \\ &\leq \frac{1}{2} \|\nabla \tilde{\xi}\|_2^2 + \frac{1}{2} \|\partial_{x_1} \tilde{u}\|_2^2. \end{aligned}$$

Integrating by parts and noticing that $\theta^{(2)} = \Delta \xi^{(2)}$, we have

$$\begin{aligned} - \int \tilde{u} \cdot \nabla \Delta \xi^{(2)} \tilde{\xi} &= \int \tilde{u} \cdot \nabla \tilde{\xi} \Delta \xi^{(2)} \\ &\leq \frac{1}{2} \|\theta^{(2)}\|_{\infty} (\|\tilde{u}\|_2^2 + \|\nabla \tilde{\xi}\|_2^2). \end{aligned}$$

For notational convenience, we write

$$J_1 = - \int \tilde{u} \cdot \nabla u^{(2)} \cdot \tilde{u}, \quad J_2 = - \int u^{(1)} \cdot \nabla \Delta \tilde{\xi} \tilde{\xi}.$$

Clearly

$$|J_1| \leq \|\tilde{u}\|_{\infty}^{\frac{2}{p}} \int |\nabla u^{(2)}| |\tilde{u}|^{2-\frac{2}{p}} \leq \|\tilde{u}\|_{\infty}^{\frac{2}{p}} \|\nabla u^{(2)}\|_p \|\tilde{u}\|_2^{2-\frac{2}{p}}$$

We know that

$$\sup_{p \geq 2} \frac{\|\nabla u^{(2)}\|_p}{p} \leq C_0$$

for C_0 independent of p . Also we have

$$\|\tilde{u}\|_\infty \leq M \equiv C \|\tilde{u}\|_2^a \|\nabla \tilde{u}\|_q^{1-a} \text{ for any } q > 2 \text{ and } a = \frac{q-2}{2(q-1)}.$$

Therefore,

$$|J_1| \leq C_0 p M^{\frac{2}{p}} \|\tilde{u}\|_2^{2-\frac{2}{p}}.$$

Using the simple fact that a function $f(x) = xA^{\frac{2}{x}}$ has the minimum $2e \log A$, we have

$$|J_1| \leq 2C_0 (\log M - \log \|\tilde{u}\|_2) \|\tilde{u}\|_2^2.$$

Integrating by parts yields

$$\begin{aligned} J_2 &= \int u^{(1)} \cdot \nabla \tilde{\xi} \Delta \tilde{\xi} \\ &= \int u^{(1)} \cdot \nabla \tilde{\xi} \partial_k \partial_k \tilde{\xi} \\ &= - \int \partial_k u^{(1)} \cdot \nabla \tilde{\xi} \partial_k \tilde{\xi} - \int u^{(1)} \cdot \nabla \partial_k \tilde{\xi} \partial_k \tilde{\xi} \\ &= - \int \partial_k u^{(1)} \cdot \nabla \tilde{\xi} \partial_k \tilde{\xi}. \end{aligned}$$

Therefore,

$$\begin{aligned} |J_2| &\leq \int |\nabla u^{(1)}| |\nabla \tilde{\xi}|^2 \\ &\leq \|\nabla \tilde{\xi}\|_\infty^{\frac{2}{p}} \int |\nabla u^{(1)}| |\nabla \tilde{\xi}|^{2-\frac{2}{p}}. \end{aligned}$$

Since $\|\nabla \tilde{\xi}\|_{\infty} \leq M$,

$$\begin{aligned} |J_2| &\leq C p M^{\frac{2}{p}} \|\nabla \tilde{\xi}\|_2^{2-\frac{2}{p}} \\ &\leq C (\log M - \log \|\nabla \tilde{\xi}\|_2) \|\nabla \tilde{\xi}\|_2^2. \end{aligned}$$

Combining the estimates allows us to conclude that

$X(t) = \|\tilde{u}\|_2^2 + \|\nabla \tilde{\xi}\|_2^2$ satisfies

$$\frac{d}{dt} X + \|\partial_x \tilde{u}\|_2^2 \leq C X + C (\log M - \log X) X,$$

where C 's are constants.

$$\frac{d}{dt}X - CX \leq C(\log M - \log X)X$$

$$\frac{d}{dt}(e^{-Ct}X) \leq Ce^{-Ct}(\log M - \log X)X$$

$$X(t) \leq e^{Ct}X(0) + C \int_0^t e^{C(t-\tau)}(\log M - \log X)X d\tau.$$

Since $X(0) = 0$, we have $X(t) \equiv 0$ for any $t > 0$ by Osgood inequality (see Lemma 1.4 below). This completes the proof of Theorem 1. \square

Lemma 1.4 (Osgood Inequality)

Let $\rho_0 \geq 0$ be a constant and $\alpha(t) \geq 0$ be a continuous function. Assume

$$\rho(t) \leq \rho_0 + \int_0^t \alpha(\tau) w(\rho(\tau)) d\tau,$$

where w satisfies

$$\int_1^\infty \frac{1}{w(r)} dr = \infty.$$

Then $\rho_0 = 0$ implies $\rho \equiv 0$, and $\rho_0 > 0$ implies that

$$-\Omega(\rho(t)) + \Omega(\rho_0) \leq \int_0^t \alpha(\tau) d\tau,$$

where $\Omega(\rho) = \int_\rho^1 \frac{dr}{w(r)}$.

In particular, the Osgood inequality applies to $w(\rho) = \rho \log \frac{M}{\rho}$ since

$$\int_1^\infty \frac{1}{r(\log M - \log r)} dr = \infty.$$

Horizontal thermal diffusion

We work on the 2D Boussinesq equations with only horizontal diffusion

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p + \theta \vec{e}_2, \\ \partial_t \theta + (u \cdot \nabla) \theta = \partial_{x_1 x_1} \theta, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x), \end{cases} \quad (1.8)$$

where we have set the coefficient of $\partial_{x_1 x_1} \theta$ to be 1, without loss of generality. (1.8) still possesses a unique global solution when (u_0, θ_0) is in a suitable functional setting. The following Theorem is coming from the following reference:

- Danchin and Paicu, M³AS (2011).

Theorem 3

Assume $u_0 \in H^1$, $\omega_0 = \nabla \times u_0 \in L^\infty$, $\theta_0 \in H^1 \cap L^\infty$, $|\partial_{x_1}|^{1+s}\theta_0 \in L^2$ for $s \in (0, \frac{1}{2}]$. Then (1.8) has a unique global solution (u, θ) satisfying

$$u \in C([0, \infty); H^1), \quad \omega \in L_{loc}^\infty([0, \infty); L^\infty),$$

$$\theta \in C([0, \infty); H^1 \cap L^\infty),$$

$$|\partial_{x_1}|^{1+s}\theta \in L_{loc}^\infty([0, \infty); L^2), \quad |\partial_{x_1}|^{2+s}\theta \in L_{loc}^2([0, \infty); L^2).$$

Here $|\partial_{x_1}|^\beta$ with $\beta \in \mathbb{R}$ is defined in terms of its Fourier transform,

$$|\partial_{x_1}|^\beta f(x) = \int e^{ix \cdot \xi} |\xi_1|^\beta \hat{f}(\xi_1, \xi_2) d\xi_1 d\xi_2.$$

The equations give us for free the global bounds on $\|\theta\|_{L^2 \cap L^\infty}$ and $\|u\|_{H^1}$. The first real step is to obtain a global bound for $\|\nabla \theta\|_2$. This can be established by writing the nonlinear term explicitly in terms of the partial derivatives in different directions and fully take advantage of the dissipation in the x -direction. The resulting estimate is given by

$$\frac{d}{dt} \|\nabla \theta\|_{L^2}^2 + \|\partial_{x_1} \nabla \theta\|_{L^2}^2 \leq B(t) \|\nabla \theta\|_{L^2}^2,$$

where $B(t)$ is integrable on $[0, \infty)$. Some special consequences of this inequality are

$$\|\omega(\cdot, T)\|_{L^4} \leq C(T), \quad \|u(\cdot, T)\|_{L^\infty} \leq C(T),$$

where $T > 0$ is an arbitrarily fixed and $C(T)$ depends on the initial data and T .

In fact, by the Sobolev embedding inequality

$$\|\partial_{x_1}\theta\|_{L^4} \leq C\|\partial_{x_1}\theta\|_{L^2}^{1/2}\|\nabla\partial_{x_1}\theta\|_{L^2}^{1/2},$$

we have

$$\int_0^T \|\partial_{x_1}\theta\|_{L^4}^4 dt \leq C \sup_{t \in [0, T]} \|\partial_{x_1}\theta\|_{L^2}^2 \int_0^T \|\partial_{x_1}\nabla\theta\|_{L^2}^2 dt < \infty.$$

It then follows from the vorticity equation that

$$\frac{1}{4} \frac{d}{dt} \|\omega\|_{L^4}^4 = \int \partial_{x_1}\theta \omega |\omega|^2 dx \leq \|\partial_{x_1}\theta\|_{L^4} \|\omega\|_{L^4}^3.$$

Therefore,

$$\begin{aligned}\|\omega\|_{L^4} &\leq \|\omega_0\|_{L^4} + \int_0^t \|\partial_{x_1}\theta\|_{L^4} d\tau \\ &\leq \|\omega_0\|_{L^4} + \left(\int_0^t \|\partial_{x_1}\theta\|_{L^4}^4 d\tau \right)^{1/4} t^{3/4}.\end{aligned}$$

Thus,

$$\|u\|_{L^\infty} \leq \|u\|_{L^2}^{1/3} \|\nabla u\|_{L^4}^{2/3} \leq C \|u\|_{L^2}^{1/3} \|\omega\|_{L^4}^{2/3} < \infty.$$

Next we show that, for $s \in (0, 1/2]$,

$$|\partial_{x_1}|^{1+s}\theta \in L_{loc}^\infty([0, \infty); L^2), \quad |\partial_{x_1}|^{2+s}\theta \in L_{loc}^2([0, \infty); L^2). \quad (1.9)$$

Clearly, $|\partial_{x_1}|^{1+s}\theta$ satisfies

$$\partial_t |\partial_{x_1}|^{1+s}\theta - \partial_{x_1 x_1} |\partial_{x_1}|^{1+s}\theta = -|\partial_{x_1}|^{1+s}(u \cdot \nabla \theta).$$

Taking the inner product with $|\partial_{x_1}|^{1+s}\theta$, we find

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \| |\partial_{x_1}|^{1+s}\theta \|_{L^2}^2 + \| |\partial_{x_1}|^{2+s}\theta \|_{L^2}^2 \\ &= - \int |\partial_{x_1}|^{1+s}(u \cdot \nabla \theta) |\partial_{x_1}|^{1+s}\theta \, dx \\ &= - \int |\partial_{x_1}|(u \cdot \nabla \theta) |\partial_{x_1}|^{1+2s}\theta \, dx \\ &= \int \frac{\partial_{x_1}}{|\partial_{x_1}|} (\partial_{x_1} u \cdot \nabla \theta + u \cdot \nabla \partial_{x_1} \theta) |\partial_{x_1}|^{1+2s}\theta \, dx. \end{aligned}$$

Since $s \in (0, 1/2]$,

$$\begin{aligned}
 & \int \frac{\partial_{x_1}}{|\partial_{x_1}|} (u \cdot \nabla \partial_{x_1} \theta) |\partial_{x_1}|^{1+2s} \theta dx \\
 & \leq \|u\|_{L^\infty} \|\nabla \partial_{x_1} \theta\|_{L^2} \| |\partial_{x_1}|^{1+2s} \theta \|_{L^2} \\
 & \leq \|u\|_{L^\infty} \|\partial_{x_1} \nabla \theta\|_{L^2}^2.
 \end{aligned}$$

Writing $\partial_{x_1} u \cdot \nabla \theta = \partial_{x_1} u_1 \partial_{x_1} \theta + \partial_{x_1} u_2 \partial_{x_2} \theta$, we have

$$\begin{aligned}
 \int \frac{\partial_{x_1}}{|\partial_{x_1}|} (\partial_{x_1} u \cdot \nabla \theta) |\partial_{x_1}|^{1+2s} \theta dx & \leq \|\omega\|_{L^4} \|\partial_{x_1} \theta\|_{L^4} \| |\partial_{x_1}|^{1+2s} \theta \|_{L^2} \\
 & \quad + \int \frac{\partial_{x_1}}{|\partial_{x_1}|} \partial_{x_1} u_2 \partial_{x_2} \theta |\partial_{x_1}|^{1+2s} \theta.
 \end{aligned}$$

In addition,

$$\begin{aligned}
 \int \frac{\partial_{x_1}}{|\partial_{x_1}|} \partial_{x_1} u_2 \partial_{x_2} \theta |\partial_{x_1}|^{1+2s} \theta &= - \int \frac{\partial_{x_1}}{|\partial_{x_1}|} u_2 \partial_{x_1} \partial_{x_2} \theta |\partial_{x_1}|^{1+2s} \theta dx \\
 &\quad - \int \frac{\partial_{x_1}}{|\partial_{x_1}|} |\partial_{x_1}|^s (u_2 \partial_{x_2} \theta) |\partial_{x_1}|^{2+s} \theta dx \\
 &\leq \|u_2\|_{L^\infty} \|\partial_{x_1} \nabla \theta\|_{L^2} \| |\partial_{x_1}|^{1+2s} \theta \|_{L^2} \\
 &\quad + \| |\partial_{x_1}|^s (u_2 \partial_{x_2} \theta) \|_{L^2} \| |\partial_{x_1}|^{2+s} \theta \|_{L^2} \\
 &\leq \|u_2\|_{L^\infty}^2 \|\partial_{x_1} \nabla \theta\|_{L^2}^2 + \frac{1}{4} \| |\partial_{x_1}|^{2+s} \theta \|_{L^2}^2 \\
 &\quad + C \|u\|_{H^1}^2 (\|\partial_2 \theta\|_{L^2} + \|\partial_1 \partial_2 \theta\|_{L^2})^2,
 \end{aligned}$$

the last inequality have used the following inequality

$$\|fg\|_{L_{x_2}^2(H_{x_1}^{1/2})} \leq C \|f\|_{H^1} (\|g\|_{L^2} + \|\partial_1 g\|_{L^2}).$$

Combining the estimates yield, for some $g \in L^1_{loc}([0, \infty))$,

$$\frac{d}{dt} \| |\partial_{x_1}|^{1+s} \theta \|_{L^2}^2 + \| |\partial_{x_1}|^{2+s} \theta \|_{L^2}^2 \leq g(t).$$

We thus have obtained (1.9). A special consequence is that

$$\omega \in L^\infty_{loc}([0, \infty); L^\infty).$$

This can be obtained by combining

$$\|\omega\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} + \int_0^t \|\partial_{x_1} \theta\|_{L^\infty} d\tau$$

with the simple estimate of the following lemma.

Lemma 4

If $\frac{1}{s_1} + \frac{1}{s_2} < 2$, $s_1 > 0$, $s_2 > 0$, then

$$\|f\|_{L^\infty} \leq C(\|f\|_{L^2} + \| |\partial_{x_1}|^{s_1} f \|_{L^2} + \| |\partial_{x_2}|^{s_2} f \|_{L^2}).$$

Applying this lemma with $s_1 = 1 + s$, $s_2 = 1$, we have

$$\|\omega\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} + C \int_0^t (\|\partial_{x_1} \theta\|_{L^2} + \| |\partial_{x_1}|^{2+s} \theta \|_{L^2} + \|\partial_{x_1} \partial_{x_2} \theta\|_{L^2}) d\tau < \infty.$$

Trivially, interpolating between L^2 and L^∞ yields $\omega \in L^q$ for $q \in [2, \infty)$. More importantly,

$$\nabla u \in L_{loc}^\infty([0, \infty); L) \quad \text{or} \quad \sup_{t \in [0, T]} \sup_{q \geq 2} \frac{\|\nabla u\|_{L^q}}{q} \leq C(T).$$

This completes the part for the existence and regularity part.

Finally we show the uniqueness. This is a consequence of the Yudovich type argument. Let $(u^{(1)}, \theta^{(1)})$ and $(u^{(2)}, \theta^{(2)})$ be two solutions. Then $\tilde{u} = u^{(1)} - u^{(2)}$ and $\tilde{\theta} = \theta^{(1)} - \theta^{(2)}$ satisfy

$$\begin{cases} \partial_t \tilde{u} + u^{(1)} \cdot \nabla \tilde{u} + \tilde{u} \cdot \nabla u^{(2)} = \theta \vec{e}_2, \\ \partial_t \tilde{\theta} + u^{(1)} \cdot \nabla \tilde{\theta} + \tilde{u} \cdot \nabla \theta^{(2)} = \partial_{x_1 x_1} \tilde{\theta}. \end{cases}$$

The most difficult term we would encounter in the further estimates is

$$\int \tilde{u}_2 \partial_{x_2} \theta^{(2)} \tilde{\theta} dx.$$

One way to handle it is to use the divergence free $\operatorname{div} u = 0$ and the identity

$$\tilde{u}_2 = (I - \partial_{x_2}^2)^{-1} \tilde{u}_2 - (I - \partial_{x_2}^2)^{-1} \partial_{x_2} \partial_{x_1} \tilde{u}_1.$$

We shall omit further details. This completes the proof of Theorem 3.

Vertical dissipation and vertical thermal diffusion

This section focuses on the global regularity problem for the 2D Boussinesq equations with vertical dissipation and vertical thermal diffusion. More precisely, we study the global existence and uniqueness of solutions to the initial-value problem of

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu \partial_{yy} u + \theta \vec{e}_2 \\ \partial_t \theta + u \cdot \nabla \theta = \kappa \partial_{yy} \theta \\ \nabla \cdot u = 0 \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x). \end{cases} \quad (1.10)$$

The global existence and uniqueness of (1.10) were established in [Cao, Chongsheng; Wu, Jiahong](#), Global regularity for the 2D anisotropic Boussinesq equations with vertical dissipation. ARMA.(2013).

The main results can be stated as follows.

Theorem 5 (Cao, Chongsheng; Wu, Jiahong, 2013)

Consider the IVP for the anisotropic Boussinesq equations with vertical dissipation (1.10). Let $\nu > 0$ and $\kappa > 0$. Let $(u_0, v_0, \theta_0) \in H^2(\mathbb{R}^2)$. Then, for any $T > 0$, (1.10) has a unique classical solution (u, v, θ) on $[0, T]$ satisfying

$$(u, v, \theta) \in C([0, T]; H^2(\mathbb{R}^2)).$$

The rest of this sections proves this theorem. As we mentioned before, the local existence and uniqueness is not very hard to obtain. Therefore, our effort will be devoted to proving the global *a priori* H^2 bounds for the solution.

We briefly explain the difficulty and point out a simple fact. The L^2 -estimates are easy to get.

$$\|\theta(t)\|_2^2 + 2\kappa \int_0^T \|\partial_y \theta\|_2^2 dt \leq \|\theta_0\|_2^2,$$

$$\|u(t)\|_2^2 + 2\nu \int_0^T \|\partial_y u\|_2^2 dt \leq (\|u_0\|_2 + t \|\theta_0\|_2)^2.$$

But the global H^1 -bound is hard to prove. Consider the equation for $\omega = \vec{\nabla} \times \vec{u}$, which satisfies

$$\partial_t \omega + \vec{u} \cdot \nabla \omega = \nu \partial_{yy} \omega + \partial_x \theta.$$

As a simple consequence,

$$\frac{1}{2} \frac{d}{dt} \int \omega^2 + \nu \int (\partial_y \omega)^2 = \int \partial_x \theta \omega.$$

In contrast to the horizontal dissipation case, the dissipation in the y -direction does not allow us to hide $\partial_x \theta$. Therefore, if we want to obtain a global bound for $\|\omega\|_2$, then we need to combine it with the estimate of $\nabla \theta$.

$$\frac{1}{2} \frac{d}{dt} \int |\nabla \theta|^2 + \kappa \int |\partial_y \nabla \theta|^2 = - \int \nabla \theta \cdot \nabla u \cdot \nabla \theta. \quad (1.11)$$

To make use of the dissipation in the y -direction, we write

$$\begin{aligned}\nabla\theta \cdot \nabla u \cdot \nabla\theta &= \partial_x u (\partial_x \theta)^2 + \partial_x v \partial_x \theta \partial_y \theta \\ &\quad + \partial_y u \partial_x \theta \partial_y \theta + \partial_y v (\partial_y \theta)^2.\end{aligned}$$

To bound the terms on the right, we need the following lemma.

Lemma 1.5

Assume that $f, g, g_y, h, h_x \in L^2(\mathbb{R}^2)$. Then

$$\iint |f g h| \, dx dy \leq C \|f\|_2 \|g\|_2^{\frac{1}{2}} \|g_y\|_2^{\frac{1}{2}} \|h\|_2^{\frac{1}{2}} \|h_x\|_2^{\frac{1}{2}}. \quad (1.12)$$

This lemma allows us to bound some of the terms suitably. For example,

$$\begin{aligned} \left| \int \partial_y u \partial_x \theta \partial_y \theta \right| &\leq C \|\partial_y u\|_2 \|\partial_x \theta\|_2^{\frac{1}{2}} \|\partial_{xy} \theta\|_2^{\frac{1}{2}} \|\partial_y \theta\|_2^{\frac{1}{2}} \|\partial_{xy} \theta\|_2^{\frac{1}{2}} \\ &\leq \frac{\kappa}{4} \|\partial_{xy} \theta\|_2^2 + C(\kappa) \|\partial_y u\|_2^2 \|\partial_x \theta\|_2 \|\partial_y \theta\|_2. \end{aligned}$$

Integrating by parts, we have

$$\begin{aligned} \int \partial_x v \partial_x \theta \partial_y \theta &= - \int \theta (\partial_x v \partial_{xy} \theta + \partial_{xy} v \partial_x \theta) \\ &\leq \|\theta_0\|_\infty \|\partial_x v\|_2 \|\partial_{xy} \theta\|_2 + \|\theta_0\|_\infty \|\partial_x \theta\|_2 \|\partial_{xy} v\|_2 \end{aligned}$$

However, the term $\int \partial_x u (\partial_x \theta)^2$ can not be bounded suitably. But if we know

$$\int_0^T \|v(t)\|_{L^\infty}^2 dt < \infty, \quad (1.13)$$

then we have, after integration by parts,

$$\begin{aligned} \int \partial_x u (\partial_x \theta)^2 &= - \int \partial_y v (\partial_x \theta)^2 = \int v \partial_x \theta \partial_{xy} \theta \\ &\leq \frac{\kappa}{4} \|\partial_{xy} \theta\|_2^2 + C(\kappa) \|v(t)\|_{L^\infty}^2 \|\partial_x \theta\|_2^2. \end{aligned}$$

Inserting the estimates above in (53) and (1.11), we are able to conclude that, if (1.13) holds, then

$$\|\omega\|_2^2 + \|\nabla\theta\|_2^2 + \nu \int (\partial_y \omega)^2 + \kappa \int |\partial_y \nabla \theta|^2 \leq C(T).$$

Unfortunately it appears to be extremely hard to prove (1.13). Therefore, we have to solve this problem through a different route.

This section presents the key ingredients in the proof as well as the proof of Theorem 5.

Proposition 1.6

Assume $(u_0, v_0, \theta_0) \in H^2$. Let (u, v, θ) be the corresponding classical solution of (1.10). Then the quantity

$$Y(t) = \|\omega\|_{H^1}^2 + \|\theta\|_{H^2}^2 + \|\omega^2 + |\nabla\theta|^2\|_2^2$$

satisfies

$$\begin{aligned} \frac{d}{dt} Y(t) + \|\omega_y\|_{H^1}^2 + \|\theta_y\|_{H^2}^2 + \int (\omega^2 + |\nabla\theta|^2) (\omega^2 + |\nabla\theta_y|^2) \\ \leq C (1 + \|\theta_0\|_\infty^2 + \|v\|_\infty^2 + \|u_y\|_2^2 + (1 + \|u\|_2^2) \|v_y\|_2^2) Y(t), \end{aligned}$$

where C is a constant.

As a special consequence of this differential inequality, we conclude that, if

$$\int_0^T \|v(t)\|_{L^\infty}^2 dt < \infty,$$

then $Y(t) < +\infty$ on $[0, T]$.

Proposition 1.7

Let $(u_0, v_0, \theta_0) \in H^2(\mathbb{R}^2)$ and let (u, v, θ) be the corresponding classical solution of (1.10). Then,

$$\sup_{r \geq 2} \frac{\|v(t)\|_{L^{2r}}}{\sqrt{r \log r}} \leq \sup_{r \geq 2} \frac{\|v_0\|_{L^{2r}}}{\sqrt{r \log r}} + B(t), \quad (1.14)$$

where $B(t)$ is an explicit integrable function of $t \in [0, \infty)$ that depends on ν, κ and the initial norm $\|(u_0, v_0, \theta_0)\|_{H^2}$.

The proof of this proposition is very delicate and provided in another subsection.

Proposition 1.8

Let $s > 1$ and $f \in H^s(\mathbb{R}^2)$. Assume that

$$\sup_{r \geq 2} \frac{\|f\|_r}{\sqrt{r \log r}} < \infty.$$

Then there exists a constant C depending on s only such that

$$\|f\|_{L^\infty(\mathbb{R}^2)} \leq C \sup_{r \geq 2} \frac{\|f\|_r}{\sqrt{r \log r}} \left[\log(e + \|f\|_{H^s(\mathbb{R}^2)}) \log \log(e + \|f\|_{H^s(\mathbb{R}^2)}) \right]^{\frac{1}{2}}. \quad (1.15)$$

Proof of Theorem 5: Applying Proposition 1.7 and using the simple fact that $\|v\|_{H^2}^2 \leq \|\omega\|_{H^1}^2 \leq Y(t)$, we obtain

$$\frac{d}{dt} Y(t) \leq A(t) Y(t) + C B^2(t) Y(t) \log(e + Y(t)) \log \log(e + Y(t)),$$

where $A(t) = C (1 + \|\theta_0\|_\infty^2 + \|u_y\|_2^2 + (1 + \|u\|_2^2) \|v_y\|_2^2)$. An application of Gronwall's inequality then concludes the proof of Theorem 5.

Proof of Proposition 1.7

Before we provide the real proof, we would like to understand how a bound of this level can be obtained. For this purpose, we make the ansatz

$$\int_0^T \|p\|_\infty^2 dt < \infty.$$

Then we show that

$$\|v\|_{L^{2r}} \leq C\sqrt{r}.$$

Recall the equation for the velocity field

$$\begin{cases} u_t + uu_x + vu_y = -p_x + \nu u_{yy}, \\ v_t + uv_x + vv_y = -p_y + \nu v_{yy} + \theta \end{cases} \quad (1.16)$$

Taking the inner product of the second equation in (1.16) with $v |v|^{2r-2}$ and integrating by parts, we obtain

$$\begin{aligned} \frac{1}{2r} \frac{d}{dt} \int |v|^{2r} + \nu(2r-1) \int v_y^2 |v|^{2r-2} \\ = (2r-1) \int p v_y |v|^{2r-2} + \int \theta v |v|^{2r-2}. \end{aligned}$$

The last term is easy to handle and we focus on the term involving the pressure p .

$$\begin{aligned} (2r-1) \int p v_y |v|^{2r-2} &= (2r-1) \int p |v|^{r-1} v_y |v|^{r-1} \\ &\leq (2r-1) \|p\|_\infty \| |v|^{r-1} \|_2 \|v_y |v|^{r-1}\|_2 \\ &\leq \frac{\nu(2r-1)}{4} \|v_y |v|^{r-1}\|_2^2 + C(2r-1) \|p\|_\infty^2 \|v\|_{2r-2}^{2r-2} \\ &\leq \frac{\nu(2r-1)}{4} \|v_y |v|^{r-1}\|_2^2 + C(2r-1) \|p\|_\infty^2 \|v\|_2^{\frac{2}{r-1}} \|v\|_{2r}^{2r-2-\frac{2}{r-1}}. \end{aligned}$$

Then, if $\int_0^T \|p\|_\infty^2 dt < \infty$,

$$\frac{1}{2r} \frac{d}{dt} \|v\|_{2r}^{2r} \leq C (2r - 1) \|p\|_\infty^2 \|v\|_2^{\frac{2}{r-1}} \|v\|_{2r}^{2r-2-\frac{2}{r-1}}$$

would yield $\|v\|_{2r} \leq C \sqrt{r}$. But unfortunately, we do not know if $\int_0^T \|p\|_\infty^2 dt < \infty$. What we can show is the following bound.

Proposition 1.9

Let $(u_0, v_0, \theta_0) \in H^2(\mathbb{R}^2)$ and let (u, v, θ) be the corresponding classical solution of (1.10). Then

$$\|(u(t), v(t))\|_4^4 + \nu \int_0^t \| |(u_y(\tau), v_y(\tau))| |(u(\tau), v(\tau)) \|_2^2 d\tau \leq M_1(t), \quad (1.17)$$

$$\|p(\cdot, t)\|_2 \leq M_2(t), \quad \int_0^t \|\nabla p(\cdot, \tau)\|_2^2 d\tau \leq M_3(t), \quad (1.18)$$

where M_1, M_2 and M_3 are explicit smooth functions of $t \in [0, \infty)$ that depend on ν, κ and the initial norm $\|(u_0, v_0, \theta_0)\|_{H^2}$.

We also need two lemmas.

Lemma 1.10

Let $f \in H^1(\mathbb{R}^2)$. Let $R > 0$. Denote by $B(0, R)$ the ball centered at zero with radius R and by $\chi_{B(0, R)}$ the characteristic function on $B(0, R)$. Write $f = \bar{f} + \tilde{f}$ with

$$\bar{f} = \mathcal{F}^{-1}(\chi_{B(0, R)} \mathcal{F}f) \quad \text{and} \quad \tilde{f} = \mathcal{F}^{-1}((1 - \chi_{B(0, R)}) \mathcal{F}f). \quad (1.19)$$

Then we have the following estimates for \bar{f} and \tilde{f} .

- (1) There exists a pure constant C independent of f and R such that

$$\|\bar{f}\|_{L^\infty(\mathbb{R}^2)} \leq C \sqrt{\log R} \|f\|_{H^1(\mathbb{R}^2)}. \quad (1.20)$$

Lemma 1.10

(2) For any $2 \leq q < \infty$, there is a constant independent of q , R and f such that

$$\|\tilde{f}\|_{L^q(\mathbb{R}^2)} \leq C \frac{q}{R^{\frac{2}{q}}} \|\tilde{f}\|_{H^1(\mathbb{R}^2)} \leq C \frac{q}{R^{\frac{2}{q}}} \|f\|_{H^1(\mathbb{R}^2)} \quad (1.21)$$

In particular, for $q = 4$,

$$\|\tilde{f}\|_{L^4(\mathbb{R}^2)} \leq \frac{C}{\sqrt{R}} \|f\|_{H^1(\mathbb{R}^2)}.$$

Lemma 1.11

Let $q \in [2, \infty)$. Assume that $f, g, g_y, h_x \in L^2(\mathbb{R}^2)$ and $h \in L^{2(q-1)}(\mathbb{R}^2)$. Then

$$\iint_{\mathbb{R}^2} |f g h| \, dx dy \leq C \|f\|_2 \|g\|_2^{1-\frac{1}{q}} \|g_y\|_2^{\frac{1}{q}} \|h\|_{2(q-1)}^{1-\frac{1}{q}} \|h_x\|_2^{\frac{1}{q}}. \quad (1.22)$$

where C is a constant depending on q only. Two special cases of (1.22) are

$$\iint |f g h| \, dx dy \leq C \|f\|_2 \|g\|_2^{\frac{2}{3}} \|g_y\|_2^{\frac{1}{3}} \|h\|_4^{\frac{2}{3}} \|h_x\|_2^{\frac{1}{3}} \quad (1.23)$$

and

$$\iint |f g h| \, dx dy \leq C \|f\|_2 \|g\|_2^{\frac{1}{2}} \|g_y\|_2^{\frac{1}{2}} \|h\|_2^{\frac{1}{2}} \|h_x\|_2^{\frac{1}{2}}. \quad (1.24)$$

Proof of Proposition 1.7.

$$\begin{cases} u_t + uu_x + vu_y = -p_x + \nu u_{yy}, \\ v_t + uv_x + vv_y = -p_y + \nu v_{yy} + \theta \end{cases} \quad (1.25)$$

Taking the inner product of the second equation in (1.25) with $v |v|^{2r-2}$ and integrating by parts, we obtain

$$\begin{aligned} & \frac{1}{2r} \frac{d}{dt} \int |v|^{2r} + \nu(2r-1) \int v_y^2 |v|^{2r-2} \\ &= (2r-1) \int p v_y |v|^{2r-2} + \int \theta v |v|^{2r-2} \\ &= (2r-1) \int \bar{p} v_y |v|^{2r-2} + (2r-1) \int \tilde{p} v_y |v|^{2r-2} + \int \theta v |v|^{2r-2}. \end{aligned}$$

By Hölder's inequality,

$$\begin{aligned} \int \theta v |v|^{2r-2} &\leq \|\theta\|_{2r} \|v\|_{2r}^{2r-1}, \\ \int \bar{p} v_y |v|^{2r-2} &\leq \|\bar{p}\|_{\infty} \|v^{r-1}\|_2 \|v_y v^{r-1}\|_2. \end{aligned} \quad (1.26)$$

Applying Lemma 1.11, we have

$$\int \tilde{p} v_y |v|^{2r-2} \leq C \|\tilde{p}\|_4^{\frac{2}{3}} \|\tilde{p}_x\|_2^{\frac{1}{3}} \|v^{r-1}\|_2^{\frac{2}{3}} \|(r-1)v_y v^{r-2}\|_2^{\frac{1}{3}} \|v_y v^{r-1}\|_2.$$

Furthermore, by Hölder's inequality,

$$\begin{aligned}\| |v|^{r-1} \|_2 &= \| v \|_{2(r-1)}^{r-1} \leq \| v \|_2^{\frac{1}{r-1}} \| v \|_{2r}^{\frac{r(r-2)}{r-1}}, \\ \| |v|^{r-2} v_y \|_2^2 &= \int |v|^{2(r-2)} v_y^2 = \int |v|^{2(r-2)} v_y^{\frac{2(r-2)}{r-1}} v_y^{\frac{2}{r-1}} \leq \| v_y \|_2^{\frac{2}{r-1}} \| v_y |v|^{r-1} \|_2^{\frac{2(r-2)}{r-1}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int \bar{p} v_y |v|^{2r-2} &\leq C \| \bar{p} \|_\infty \| v \|_2^{\frac{1}{r-1}} \| v \|_{2r}^{\frac{r(r-2)}{r-1}} \| v_y v^{r-1} \|_2, \\ \int \tilde{p} v_y |v|^{2r-2} &\leq C (r-1)^{\frac{1}{3}} \| \tilde{p} \|_4^{\frac{2}{3}} \| \tilde{p}_x \|_2^{\frac{1}{3}} \| v \|_2^{\frac{2}{3(r-1)}} \| v \|_{2r}^{\frac{2r(r-2)}{3(r-1)}} \\ &\quad \times \| v_y \|_2^{\frac{1}{3(r-1)}} \| v_y |v|^{r-1} \|_2^{1+\frac{(r-2)}{3(r-1)}}.\end{aligned}$$

By Young's inequality and Lemma 1.10,

$$(2r-1) \int \bar{p} v_y |v|^{2r-2} \leq \frac{\nu}{4} (2r-1) \|v_y v^{r-1}\|_2^2 \\ + C(2r-1)(\log R) \|p\|_{H^1}^2 \|v\|_2^{\frac{2}{r-1}} \|v\|_{2r}^{2r-2-\frac{2}{r-1}}.$$

By Young's inequality and Lemmas 1.10,

$$(2r-1) \int \tilde{p} v_y |v|^{2r-2} \leq \frac{\nu}{4} (2r-1) \|v_y v^{r-1}\|_2^2 + C(2r-1)(r-1)^{\frac{2r-2}{2r-1}} \\ \times \|\tilde{p}\|_4^{\frac{4(r-1)}{2r-1}} \|\tilde{p}_x\|_2^{\frac{2(r-1)}{2r-1}} \|v_y\|_2^{\frac{2}{2r-1}} \|v\|_2^{\frac{4}{2r-1}} \|v\|_{2r}^{2r-2-\frac{2(r+1)}{2r-1}} \\ \leq \frac{\nu}{4} (2r-1) \|v_y v^{r-1}\|_2^2 + C(2r-1)(r-1)^{\frac{2r-2}{2r-1}} R^{-\frac{r-1}{2r-1}} \\ \times \|p\|_{L^4}^{\frac{2r-2}{2r-1}} \|p\|_{H^1}^{\frac{4r-4}{2r-1}} \|v_y\|_2^{\frac{2}{2r-1}} \|v\|_2^{\frac{4}{2r-1}} \|v\|_{2r}^{2r-3-\frac{3}{2r-1}}.$$

Without loss of generality, we assume $\|v\|_{2r} \geq 1$. Inserting (1.26), (1.27) and (1.27) in (1.26), we have

$$\begin{aligned} & \frac{1}{2r} \frac{d}{dt} \|v\|_{L^{2r}}^{2r} + \frac{\nu}{2} (2r-1) \int v_y^2 |v|^{2r-2} dx \\ & \leq C(2r-1)(\log R) \|p\|_{H^1}^2 \|v\|_2^{\frac{2}{r-1}} \|v\|_{2r}^{2r-2} \\ & + C(2r-1)(r-1)^{\frac{2r-2}{2r-1}} R^{-\frac{r-1}{2r-1}} \|p\|_{L^4}^{\frac{2r-2}{2r-1}} \|p\|_{H^1}^{\frac{4r-4}{2r-1}} \|v_y\|_2^{\frac{2}{2r-1}} \|v\|_2^{\frac{4}{2r-1}} \|v\|_{2r}^{2r-2} \\ & + \|\theta\|_{L^{2r}} \|v\|_{L^{2r}}^{2r-1}. \end{aligned}$$

Especially,

$$\begin{aligned} \frac{d}{dt} \|v\|_{L^{2r}}^2 &\leq C(2r-1)(\log R) \|p\|_{H^1}^2 \|v\|_2^{\frac{2}{r-1}} \\ &\quad + C(2r-1)(r-1)^{\frac{2r-2}{2r-1}} R^{-\frac{r-1}{2r-1}} \|p\|_{L^4}^{\frac{2r-2}{2r-1}} (\|p\|_{H^1}^2 + \|v_y\|_2^2) \|v\|_2^{\frac{4}{2r-1}} \\ &\quad + \|\theta\|_{L^{2r}}^2 + \|v\|_{L^{2r}}^2. \end{aligned}$$

Taking $R = (2r-1)^{\frac{2r-1}{2r-2}}(r-1)^2$, integrating in time and applying Propositions, we obtain

$$\|v(t)\|_{L^{2r}}^2 \leq \|v_0\|_{L^{2r}}^2 + B_1(t)r \log r + B_2(t),$$

where B_1 and B_2 are explicit integrable functions. Therefore,

$$\sup_{r \geq 2} \frac{\|v(t)\|_{L^{2r}}^2}{r \log r} \leq \sup_{r \geq 2} \frac{\|v_0\|_{L^{2r}}^2}{r \log r} + (B_1(t) + B_2(t)).$$

This completes the proof of Proposition 1.7.

This subsection proves Proposition 1.8.

By the Littlewood-Paley decomposition, we can write

$$f = S_{N+1}f + \sum_{j=N+1}^{\infty} \Delta_j f,$$

where Δ_j denotes the Fourier localization operator and

$$S_{N+1} = \sum_{j=-1}^N \Delta_j.$$

The definitions of Δ_j and S_N are now standard. Therefore,

$$\|f\|_{\infty} \leq \|S_{N+1}f\|_{\infty} + \sum_{j=N+1}^{\infty} \|\Delta_j f\|_{\infty}.$$

We denote the terms on the right by I and II . By Bernstein's inequality, for any $q \geq 2$,

$$|I| \leq 2^{\frac{2N}{q}} \|S_{N+1} f\|_q \leq 2^{\frac{2N}{q}} \|f\|_q.$$

Taking $q = N$, we have

$$|I| \leq 4 \|f\|_N \leq 4 \sqrt{N \log N} \sup_{r \geq 2} \frac{\|f\|_r}{\sqrt{r \log r}}.$$

By Bernstein's inequality again, for any $s > 1$,

$$\begin{aligned} |II| &\leq \sum_{j=N+1}^{\infty} 2^j \|\Delta_j f\|_2 = \sum_{j=N+1}^{\infty} 2^{-j(s-1)} 2^{sj} \|\Delta_j f\|_2 \\ &= C 2^{-(N+1)(s-1)} \|f\|_{B_{2,2}^s}. \end{aligned}$$

where C is a constant depending on s only.

By identifying $B_{2,2}^s$ with H^s , we obtain

$$\|f\|_{\infty} \leq 4\sqrt{N \log N} \sup_{r \geq 2} \frac{\|f\|_r}{\sqrt{r \log r}} + C 2^{-(N+1)(s-1)} \|f\|_{H^s}.$$

We obtain the desired inequality (1.15) by taking

$$N = \left\lceil \frac{1}{s-1} \log_2(e + \|f\|_{H^s}) \right\rceil,$$

where $[a]$ denotes the largest integer less than or equal to a . □

Our attention here focuses on the following 2D Boussinesq equations with horizontal dissipation in the vertical velocity equation and vertical dissipation in the temperature equation

$$\left\{ \begin{array}{l} \partial_t u + u \partial_x u + v \partial_y u + \partial_x p = 0, \\ \partial_t v + u \partial_x v + v \partial_y v + \partial_y p - \partial_{xx} v = \theta, \\ \partial_t \theta + u \partial_x \theta + v \partial_y \theta - \partial_{yy} \theta = 0, \\ \partial_x u + \partial_y v = 0, \\ u(x, y, 0) = u_0(x, y), \quad v(x, y, 0) = v_0(x, y), \\ \theta(x, y, 0) = \theta_0(x, y). \end{array} \right. \quad (1.27)$$

Wu, Jiahong; Xu, Xiaojing; Ye, Zhuan, J.Math. Fluid Mech.(2017) established several global bounds, which may be useful in the eventual resolution of whether or not (1.27) is globally well-posed.

Theorem 1.12 (Wu, Jiahong; Xu, Xiaojing; Ye, Zhuan (2017))

Assume that $(\vec{u}_0, \theta_0) \in H^\sigma(\mathbb{R}^2)$ with $\sigma > 2$ and $\nabla \cdot \vec{u}_0 = 0$. Let (\vec{u}, θ) be the corresponding solution of (1.27). Then, (\vec{u}, θ) admits the following global bounds, for any $T > 0$ and $t \leq T$,

$$\|\vec{u}(t)\|_{H^1}^2 + \int_0^t \|\partial_x \nabla v(\tau)\|_{L^2}^2 d\tau \leq C,$$

where $C = C(T, \vec{u}_0, \theta_0)$;

$$\|\theta(t)\|_{H^1}^2 + \int_0^t \|\partial_y \nabla \theta(\tau)\|_{L^2}^2 d\tau \leq C,$$

where $C = C(T, \vec{u}_0, \theta_0)$;

Theorem 1.12 (Wu, Jiahong; Xu, Xiaojing; Ye, Zhuan (2017))

$$\left\| |\partial_y|^{1+s} \theta(t) \right\|_{L^2}^2 + \int_0^t \left\| |\partial_y|^{2+s} \theta(\tau) \right\|_{L^2}^2 d\tau \leq C,$$

$$\int_0^t \left\| \partial_y \theta(\tau) \right\|_{L^\infty}^2 d\tau \leq C,$$

where $0 < s < \frac{1}{2}$ and $C = C(T, \vec{u}_0, \theta_0)$;

$$\left\| \partial_x \theta(t) \right\|_{L^q} \leq C, \quad 2 \leq q < \infty$$

where $C = C(T, q, \vec{u}_0, \theta_0)$.

The following anisotropic Sobolev inequalities were mentioned before will be frequently used later.

Lemma 1.13

The following anisotropic Sobolev inequalities hold,

$$\int_{\mathbb{R}^2} |fgh| dx dy \leq C \|f\|_{L^2} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_x g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}^{\frac{1}{2}} \|\partial_y h\|_{L^2}^{\frac{1}{2}},$$

$$\int_{\mathbb{R}^2} |fgh| dx dy \leq C \|f\|_{L^2} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_x g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}^{\frac{2\gamma-1}{2\gamma}} \| |\partial_y|^\gamma h \|_{L^2}^{\frac{1}{2\gamma}}, \quad \frac{1}{2} < \gamma \leq 1$$

Proof of Theorem 1.12. To start with, let us apply the basic energy estimate to the system (1.27) to obtain

$$\begin{aligned}\|\theta(t)\|_{L^2}^2 + \int_0^t \|\partial_y \theta\|_{L^2}^2 d\tau &\leq \|\theta_0\|_{L^2}^2, \\ \|\theta(t)\|_{L^q} &\leq \|\theta_0\|_{L^q}, \quad q \in [2, \infty],\end{aligned}$$

and

$$\|\vec{u}(t)\|_{L^2} + \int_0^t \|\partial_x v\|_{L^2}^2 d\tau \leq C(t, \vec{u}_0, \theta_0).$$

Taking the inner product of the first two equations in (1.27) with $\Delta \vec{u}$ yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \vec{u}\|_{L^2}^2 + \|\partial_x \nabla v\|_{L^2}^2 &= \int_{\mathbb{R}^2} \theta \Delta v dx dy \\ &= \int_{\mathbb{R}^2} \theta \partial_{xx} v dx dy + \int_{\mathbb{R}^2} \theta \partial_{yy} v dx dy \\ &\leq \|\theta\|_{L^2} \|\partial_x \nabla v\|_{L^2} + \|\partial_y \theta\|_{L^2} \|\nabla v\|_{L^2} \\ &\leq \frac{1}{2} \|\partial_x \nabla v\|_{L^2}^2 + C \|\theta\|_{L^2}^2 + \|\partial_y \theta\|_{L^2} \|\nabla \vec{u}\|_{L^2}. \end{aligned}$$

Applying the Gronwall inequality yields

$$\|\nabla \vec{u}(t)\|_{L^2}^2 + \int_0^t \|\partial_x \nabla v\|_{L^2}^2 d\tau \leq C < \infty.$$

Taking the inner product of the third equation in (1.27) with $\Delta\theta$ leads to

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \theta(t)\|_{L^2}^2 + \|\partial_y \nabla \theta\|_{L^2}^2 &= - \int_{\mathbb{R}^2} \nabla (u \partial_x \theta + v \partial_y \theta) \cdot \nabla \theta dx dy \\ &= - \int_{\mathbb{R}^2} \partial_x u \partial_x \theta \partial_x \theta dx dy - \int_{\mathbb{R}^2} \partial_x v \partial_y \theta \partial_x \theta dx dy \\ &\quad - \int_{\mathbb{R}^2} \partial_y u \partial_x \theta \partial_y \theta dx dy - \int_{\mathbb{R}^2} \partial_y v \partial_y \theta \partial_y \theta dx dy. \end{aligned} \tag{1.28}$$

By Lemma 1.13 and $\partial_x u + \partial_y v = 0$,

$$\begin{aligned}
 \int_{\mathbb{R}^2} \partial_x u \partial_x \theta \partial_x \theta dx dy &= - \int_{\mathbb{R}^2} \partial_y v \partial_x \theta \partial_x \theta dx dy = 2 \int_{\mathbb{R}^2} v \partial_x \theta \partial_{xy} \theta dx dy \\
 &\leq C \|\partial_{xy} \theta\|_{L^2} \|\partial_x \theta\|_{L^2}^{\frac{1}{2}} \|\partial_{xy} \theta\|_{L^2}^{\frac{1}{2}} \|v\|_{L^2}^{\frac{1}{2}} \|\partial_x v\|_{L^2}^{\frac{1}{2}} \\
 &\leq \frac{1}{8} \|\partial_y \nabla \theta\|_{L^2}^2 + C \|v\|_{L^2}^2 \|\nabla \vec{u}\|_{L^2}^2 \|\nabla \theta\|_{L^2}^2
 \end{aligned} \tag{1.29}$$

The other three terms can be bounded similarly.

Then, we have

$$\frac{d}{dt} \|\nabla \theta(t)\|_{L^2}^2 + \|\partial_y \nabla \theta\|_{L^2}^2 \leq C \|\theta\|_{L^\infty}^2 \|\nabla \theta\|_{L^2}^2 + C \|v\|_{L^2}^2 \|\nabla \vec{u}\|_{L^2}^2 \|\nabla \theta\|_{L^2}^2.$$

An easy application of Gronwall's inequality gives

$$\|\nabla \theta(t)\|_{L^2}^2 + \int_0^t \|\partial_y \nabla \theta\|_{L^2}^2 d\tau < \infty.$$

Combining above estimates, one gets

$$\|\vec{u}(t)\|_{H^1}^2 + \|\theta(t)\|_{H^1}^2 + \int_0^t \left(\|\partial_x v\|_{H^1}^2 + \|\partial_y \theta\|_{H^1}^2 \right) d\tau \leq C < \infty. \quad (1.30)$$

Applying $|\partial_y|^{1+s}$ ($0 < s < \frac{1}{2}$) to the temperature equation and multiplying the resulting equation by $|\partial_y|^{1+s} \theta$, we get

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \left\| |\partial_y|^{1+s} \theta(t) \right\|_{L^2}^2 + \left\| |\partial_y|^{2+s} \theta \right\|_{L^2}^2 &= \int_{\mathbb{R}^2} |\partial_y|^{1+s} (\vec{u} \cdot \nabla \theta) |\partial_y|^{1+s} \theta dx dy \\
 &= \int_{\mathbb{R}^2} |\partial_y| (\vec{u} \cdot \nabla \theta) |\partial_y|^{1+2s} \theta dx dy \\
 &= \int_{\mathbb{R}^2} \frac{\partial_y}{|\partial_y|} (\partial_y \vec{u} \cdot \nabla \theta) |\partial_y|^{1+2s} \theta dx dy \\
 &\quad + \int_{\mathbb{R}^2} \frac{\partial_y}{|\partial_y|} (\vec{u} \cdot \nabla \partial_y \theta) |\partial_y|^{1+2s} \theta dx dy \\
 &:= I + J.
 \end{aligned}$$

By Lemma 1.13, the term J can be bounded as

$$\begin{aligned}
 J &= \int_{\mathbb{R}^2} \frac{\partial_y}{|\partial_y|} (\vec{u} \cdot \nabla \partial_y \theta) |\partial_y|^{1+2s} \theta dx dy \\
 &= \int_{\mathbb{R}^2} (\vec{u} \cdot \nabla \partial_y \theta) \frac{\partial_y}{|\partial_y|} |\partial_y|^{1+2s} \theta dx dy \\
 &\leq C \|\partial_y \nabla \theta\|_{L^2} \|\vec{u}\|_{L^2}^{\frac{1}{2}} \|\partial_x \vec{u}\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial_y}{|\partial_y|} |\partial_y|^{1+2s} \theta \right\|_{L^2}^{\frac{1-2s}{2-2s}} \left\| \frac{\partial_y}{|\partial_y|} |\partial_y|^{1+2s+(1-s)} \theta \right\|_{L^2}^{\frac{1}{2-2s}} \\
 &\leq C \|\vec{u}\|_{H^1} \|\partial_y \nabla \theta\|_{L^2}^{\frac{3-4s}{2-2s}} \left\| |\partial_y|^{2+s} \theta \right\|_{L^2}^{\frac{1}{2-2s}} \quad \left(0 < s < \frac{1}{2} \right) \\
 &\leq \frac{1}{8} \left\| |\partial_y|^{2+s} \theta \right\|_{L^2}^2 + C \|\vec{u}\|_{H^1}^{3-4s} \|\partial_y \nabla \theta\|_{L^2}^2.
 \end{aligned}$$

Now we turn to the term I , which can be written as

$$I = \int_{\mathbb{R}^2} \frac{\partial_y}{|\partial_y|} (\partial_y u \partial_x \theta + \partial_y v \partial_y \theta) |\partial_y|^{1+2s} \theta dx dy := I_1 + I_2.$$

Integrating by parts, we get

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^2} \frac{\partial_y}{|\partial_y|} (\partial_y u \partial_x \theta) |\partial_y|^{1+2s} \theta dx dy \\ &= - \int_{\mathbb{R}^2} \frac{\partial_y}{|\partial_y|} (u \partial_y \partial_x \theta) |\partial_y|^{1+2s} \theta dx dy - \int_{\mathbb{R}^2} \frac{\partial_y}{|\partial_y|} (u \partial_x \theta) \partial_y |\partial_y|^{1+2s} \theta dx dy \\ &= - \int_{\mathbb{R}^2} (u \partial_y \partial_x \theta) \frac{\partial_y}{|\partial_y|} |\partial_y|^{1+2s} \theta dx dy - \int_{\mathbb{R}^2} |\partial_y|^s (u \partial_x \theta) \frac{\partial_y}{|\partial_y|} \partial_y |\partial_y|^{1+s} \theta dx dy \\ &:= I_{11} + I_{12} \end{aligned}$$

Obviously, the term I_{11} admits the same bound as the term J , that is,

$$I_{11} \leq \frac{1}{8} \left\| |\partial_y|^{2+s} \theta \right\|_{L^2}^2 + C \|\vec{u}\|_{H^1}^{\frac{4-4s}{3-4s}} \|\partial_y \nabla \theta\|_{L^2}^2.$$

By virtue of the Sobolev embedding, the term I_{12} can be estimated as follows

$$\begin{aligned} I_{12} &\leq C \left\| |\partial_y|^s (u \partial_x \theta) \right\|_{L^2} \left\| |\partial_y|^{2+s} \theta \right\|_{L^2} \\ &\leq C \|u\|_{H^1} (\|\partial_x \theta\|_{L^2} + \|\partial_y \nabla \theta\|_{L^2}) \left\| |\partial_y|^{2+s} \theta \right\|_{L^2} \\ &\leq \frac{1}{8} \left\| |\partial_y|^{2+s} \theta \right\|_{L^2}^2 + C \|u\|_{H^1}^2 (\|\partial_x \theta\|_{L^2} + \|\partial_y \nabla \theta\|_{L^2})^2 \end{aligned}$$

where we have applied the following estimate

$$\begin{aligned}
\| |\partial_y|^s (u \partial_x \theta) \|_{L^2} &= \| (u \partial_x \theta)(x, y) \|_{L_x^2 H_y^s} \\
&\leq C \left\| \| (u \partial_x \theta)(x, y) \|_{H_y^s} \right\|_{L_x^2} \\
&\leq C \left\| \| u(x, y) \|_{H_y^{s_1}} \| \partial_x \theta(x, y) \|_{H_y^{s_2}} \right\|_{L_x^2} \\
&\quad \times \left(s_1, s_2 < \frac{1}{2}, s + \frac{1}{2} = s_1 + s_2 > 0 \right) \\
&\leq C \left\| \| u(x, y) \|_{H_y^1} \| \partial_x \theta(x, y) \|_{H_y^1} \right\|_{L_x^2} \\
&\leq C \| u(x, y) \|_{L_x^\infty H_y^1} \left\| \| \partial_x \theta(x, y) \|_{L_y^2} + \| \partial_y \partial_x \theta(x, y) \|_{L_y^2} \right\|_{L_x^2} \\
&\leq C \| u(x, y) \|_{H_x^1 H_y^1} \left(\left\| \| \partial_x \theta(x, y) \|_{L_y^2} \right\|_{L_x^2} + \left\| \| \partial_y \partial_x \theta(x, y) \|_{L_y^2} \right\|_{L_x^2} \right) \\
&= C \| u \|_{H^1} (\| \partial_x \theta \|_{L^2} + \| \partial_y \partial_x \theta \|_{L^2})
\end{aligned}$$

Therefore, it directly yields

$$I_1 \leq \frac{1}{4} \left\| |\partial_y|^{2+s} \theta \right\|_{L^2}^2 + C \|\vec{u}\|_{H^1}^{\frac{4-4s}{3-4s}} \|\partial_y \nabla \theta\|_{L^2}^2 + C \|u\|_{H^1}^2 (\|\partial_x \theta\|_{L^2} + \|\partial_y \nabla \theta\|_{L^2})^2.$$

Finally, by Young inequality, we arrive at

$$\begin{aligned} I_2 &= \int_{\mathbb{R}^2} \frac{\partial_y}{|\partial_y|} (\partial_y v \partial_y \theta) |\partial_y|^{1+2s} \theta dx dy \\ &= \int_{\mathbb{R}^2} (\partial_y v \partial_y \theta) \frac{\partial_y}{|\partial_y|} |\partial_y|^{1+2s} \theta dx dy \\ &\leq C \|\partial_y v\|_{L^2} \|\partial_y \theta\|_{L^2}^{\frac{1}{2}} \|\partial_x \partial_y \theta\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial_y}{|\partial_y|} |\partial_y|^{1+2s} \theta \right\|_{L^2}^{\frac{1-2s}{2-2s}} \left\| \frac{\partial_y}{|\partial_y|} |\partial_y|^{2+s} \theta \right\|_{L^2}^{\frac{1}{2-2s}} \\ &\leq C \|\vec{u}\|_{H^1} \|\theta\|_{H^1}^{\frac{1}{2}} \|\partial_y \nabla \theta\|_{L^2}^{\frac{2-3s}{2-2s}} \left\| |\partial_y|^{2+s} \theta \right\|_{L^2}^{\frac{1}{2-2s}} \left(0 < s < \frac{1}{2} \right) \\ &\leq \frac{1}{8} \left\| |\partial_y|^{2+s} \theta \right\|_{L^2}^2 + C \|\vec{u}\|_{H^1}^{\frac{4-4s}{3-4s}} \|\theta\|_{H^1}^{\frac{2-2s}{3-4s}} \|\partial_y \nabla \theta\|_{L^2}^{\frac{4-6s}{3-4s}}. \end{aligned}$$

Combining the estimates for I and J together, we thus conclude

$$\begin{aligned} \frac{d}{dt} \left\| |\partial_y|^{1+s} \theta(t) \right\|_{L^2}^2 + \left\| |\partial_y|^{2+s} \theta \right\|_{L^2}^2 &\leq C \|\vec{u}\|_{H^1}^{\frac{4-4s}{3-4s}} \|\partial_y \nabla \theta\|_{L^2}^2 \\ &+ C \|u\|_{H^1}^2 \left(\|\partial_x \theta\|_{L^2} + \|\partial_y \nabla \theta\|_{L^2} \right)^2 + C \|\vec{u}\|_{H^1}^{\frac{4-4s}{3-4s}} \|\theta\|_{H^1}^{\frac{2-2s}{3-4s}} \|\partial_y \nabla \theta\|_{L^2}^{\frac{4-6s}{3-4s}}, \end{aligned}$$

which, along with Gronwall's inequality, yields

$$\left\| |\partial_y|^{1+s} \theta(t) \right\|_{L^2}^2 + \int_0^t \left\| |\partial_y|^{2+s} \theta(\tau) \right\|_{L^2}^2 d\tau < \infty, \quad \text{for any } 0 < s < \frac{1}{2}. \quad (1.31)$$

The above key bound allows us to show that

$$\int_0^T \|\partial_y \theta(\tau)\|_{L^\infty}^2 d\tau < \infty. \quad (1.32)$$

Indeed, we can deduce that

$$\begin{aligned}
 \|\partial_y \theta\|_{L^\infty} &\leq \left\| \widehat{\partial_y \theta} \right\|_{L^1} \\
 &\leq \int_{\mathbb{R}^2} \left| \xi_y \widehat{\theta}(\xi_x, \xi_y) \right| d\xi_x d\xi_y \\
 &\leq \left(\int_{\mathbb{R}^2} \left(1 + |\xi_x|^2 + |\xi_y|^{2+2s} \right) |\xi_y|^2 \left| \widehat{\theta}(\xi_x, \xi_y) \right|^2 d\xi_x d\xi_y \right)^{\frac{1}{2}} \\
 &\quad \times \left(\int_{\mathbb{R}^2} \left(1 + |\xi_x|^2 + |\xi_y|^{2+2s} \right)^{-1} d\xi_x d\xi_y \right)^{\frac{1}{2}} \quad (s > 0) \\
 &\leq C \left(\|\partial_y \theta\|_{L^2} + \|\partial_y \partial_x \theta\|_{L^2} + \left\| |\partial_y|^{2+s} \theta \right\|_{L^2} \right),
 \end{aligned}$$

where we have used

$$\int_{\mathbb{R}^2} \left(1 + |\xi_x|^2 + |\xi_y|^{2+2s} \right)^{-1} d\xi_x d\xi_y = \int_{\mathbb{R}^2} \left(1 + |\xi_x|^2 \right)^{\frac{-(1+2s)}{2+2s}} \left(1 + \eta^{2+2s} \right)^{-1} d\xi_x d\eta < \infty$$

by making the change of variable $\xi_y = \left(1 + |\xi_x|^2\right)^{\frac{1}{2+2x}} \eta$. Thus, we get the desired bound (1.32).

Taking ∂_x on the temperature equation and multiplying the resulting equation by $|\partial_x \theta|^{q-2} \partial_x \theta$, we have

$$\begin{aligned}
 \frac{1}{q} \frac{d}{dt} \|\partial_x \theta\|_{L^q}^q + (q-1) \int_{\mathbb{R}^2} (\partial_y \partial_x \theta)^2 |\partial_x \theta|^{q-2} dx dy \\
 = \int_{\mathbb{R}^2} \partial_x (u \partial_x \theta + v \partial_y \theta) |\partial_x \theta|^{q-2} \partial_x \theta dx dy \\
 = \int_{\mathbb{R}^2} \partial_x u \partial_x \theta |\partial_x \theta|^{q-2} \partial_x \theta dx dy \\
 + \int_{\mathbb{R}^2} \partial_x v \partial_y \theta |\partial_x \theta|^{q-2} \partial_x \theta dx dy,
 \end{aligned} \tag{1.33}$$

where in the last line we have used the following fact due to the incompressibility of \vec{u}

$$\int_{\mathbb{R}^2} (u \partial_x \partial_x \theta + v \partial_y \partial_x \theta) |\partial_x \theta|^{q-2} \partial_x \theta dx dy = 0$$

Integration by parts and Young inequality allow us to show

$$\begin{aligned}
 \int_{\mathbb{R}^2} \partial_x u \partial_x \theta |\partial_x \theta|^{q-2} \partial_x \theta dx dy &= - \int_{\mathbb{R}^2} \partial_y v \partial_x \theta |\partial_x \theta|^{q-2} \partial_x \theta dx dy \\
 &\leq q \int_{\mathbb{R}^2} |v| |\partial_x \theta|^{q-1} |\partial_y \partial_x \theta| dx dy \\
 &\leq \frac{q-1}{2} \int_{\mathbb{R}^2} (\partial_y \partial_x \theta)^2 |\partial_x \theta|^{q-2} dx dy + Cq \int_{\mathbb{R}^2} |v|^2 |\partial_x \theta|^q dx dy \\
 &\leq \frac{q-1}{2} \int_{\mathbb{R}^2} (\partial_y \partial_x \theta)^2 |\partial_x \theta|^{q-2} dx dy + Cq \|v\|_{L^\infty}^2 \|\partial_x \theta\|_{L^q}^q.
 \end{aligned}
 \tag{1.34}$$

Invoking Sobolev interpolation and Young inequality, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} \partial_x v \partial_y \theta |\partial_x \theta|^{q-2} \partial_x \theta dx dy &\leq C \|\partial_y \theta\|_{L^\infty} \|\partial_x v\|_{L^q} \|\partial_x \theta\|_{L^q}^{q-1} \\ &\leq C \sqrt{q} \|\partial_y \theta\|_{L^\infty} \|\partial_x v\|_{L^2}^{\frac{2}{q}} \|\partial_x \nabla v\|_{L^2}^{\frac{q-2}{q}} \|\partial_x \theta\|_{L^q}^{q-1}, \end{aligned} \quad (1.35)$$

where the following interpolation has been used

$$\|f\|_{L^q} \leq C \sqrt{q} \|f\|_{L^2}^{\frac{2}{q}} \|\nabla f\|_{L^2}^{\frac{q-2}{q}}, \quad 2 \leq q < \infty,$$

for some absolute constant C independent of q .

Substituting (1.34) and (1.35) into (1.33), we immediately obtain

$$\frac{d}{dt} \|\partial_x \theta\|_{L^q} \leq Cq \|v\|_{L^\infty}^2 \|\partial_x \theta\|_{L^q} + C\sqrt{q} \|\partial_y \theta\|_{L^\infty} \|\partial_x v\|_{L^2}^{\frac{2}{q}} \|\partial_x \nabla v\|_{L^2}^{\frac{q-2}{q}}. \quad (1.36)$$

By (1.30), it follows that

$$\int_0^T \|v(\tau)\|_{L^\infty}^2 d\tau < \infty. \quad (1.37)$$

Noticing the bounds (1.30), (1.31) and (1.37), then making use of Gronwall inequality, one can deduce from the inequality (1.36) that

$$\|\partial_x \theta(t)\|_{L^q} \leq C(T, q, \vec{u}_0, \theta_0) < \infty, \quad 2 \leq q < \infty.$$

Therefore, this concludes the proof of Theorem 1.12. \square